Calculus: Series Convergence and Divergence
Notes, Examples, and Practice Questions (with Solutions)

Topics include geometric, power, and p-series, ratio and root tests, sigma notation, taylor and maclaurin series, and more.

Mathplane.com
Geometric Series
\[ \sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots \]
\(a = \) initial value
\(r = \) common ratio (growth factor)
("exponent increases; base is constant")

TEST:
- \(|r| \geq 1\) diverges
- \(|r| < 1\) converges

If the series converges, then
\[ \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \]

**Examples:**
\[ \sum_{n=0}^{\infty} \frac{8}{2^n} = 8 + 4 + 2 + \ldots \]
converges

Since \(\frac{1}{2} < 1\), it converges
\[ \frac{n}{1-r} = \frac{8}{(1-1/2)} - 16 \]
\[ \sum_{n=0}^{\infty} 3^n = .7 + 2.1 + 6.3 + \ldots \]
diverges

p-Series
\[ \sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots \]
("exponent is constant; fraction is increasing")

TEST:
- \(p \leq 1\) diverges
- \(p > 1\) converges and,
\[ \frac{1}{p-1} < \sum_{n=1}^{\infty} \frac{1}{n^p} < 1 + \frac{1}{p-1} \]

**Examples:**
\[ \sum_{n=1}^{\infty} \frac{5}{n^3} = 5 + \frac{5}{8} + \frac{5}{27} + \frac{5}{81} + \ldots \]
converges

Since \(p = 3\), it converges
\[ \sum_{n=1}^{\infty} \frac{3}{n^p} = 3 + 2.12 + 1.73 + 1.5 + \ldots \]
diverges

Since \(p = 1/2\), it diverges

(note: the sequence is converging to 0, but the series is diverging...)

Harmonic Series ("a special p-series")
\[ \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots \]

Since \(p - 1\), it diverges

Power Series (centered at a)
\[ f(x) = \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} \] where the domain of \(f\) is the set of all \(x\) for which the power series converges.
\(c\) are the 'coefficients' of each term (constants)
\(a\) is a constant
\(x\) is a variable

TEST:
- \(|x-a| < R\) converges
- \(|x-a| > R\) diverges
- \(|x-a| = R\) inconclusive

**Example:**
\[ \sum_{n=1}^{\infty} \frac{n}{4^n} (x+6)^n \] What is the interval of convergence?

Using the ratio test,
\[ L = \lim_{n \to \infty} \frac{n+1}{n} \frac{a^{n+1}}{a^n} \]
\[ = \lim_{n \to \infty} \frac{n+1}{4^n} (x+6)^n \]
\[ = \lim_{n \to \infty} \frac{n+1}{4^n} (x+6) \]
\[ = \lim_{n \to \infty} \frac{4n}{n} (x+6) \]
\[ L = |x+6| \]

If \(\frac{1}{4} |x+6| < 1\) converges
\(|x+6| < 4\), then series converges

If \(\frac{1}{4} |x+6| > 1\) diverges
\(|x+6| > 4\), then series diverges

So, the radius of convergence \(R = 4\)
and, the interval of convergence is \(-10 < x < -2\)
TEST: Sequence Test  
If \( \lim_{n \to \infty} a_n \neq 0 \), then \( \sum_{n=0}^{\infty} \) DIVERGES

*Example:*  Sequence 3, 6, 9, 12, ... is geometric

\[ a_n = 3(2)^{k-1} \quad \text{a} = 3 \quad r = 2 \quad \text{and, since} \ r > 2, \ \text{it diverges...} \]

Therefore, the series \( 3 + 6 + 9 + 12 + ... \) is diverging...

TEST: Sequence Test  
If \( \sum_{n=0}^{\infty} \) converges, then \( \lim_{n \to \infty} a_n = 0 \)

*Example:*  
\[ \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... = 2 \]

and,  \( \lim_{n \to \infty} \frac{1}{2^n} = 0 \)

NOTE: Converse isn't true... i.e. if \( \lim_{n \to \infty} a_n = 0 \) then \( \sum_{n=0}^{\infty} \) converges OR diverges...

*Example:*  Harmonic series...

\[ \sum_{n=1}^{\infty} \frac{1}{n} \quad p = 1, \ \text{so diverges} \quad \text{Series} \ P = 1 + \frac{1}{2} + \frac{1}{3} + ... \]

However,  sequence \( 1, \frac{1}{2}, \frac{1}{3}, ... \) is obviously going to 0 converges

TEST: Integral Test  
If \( \sum_{n=1}^{\infty} a_n \) converges, then \( \int_{1}^{\infty} f(x) \, dx \) converges

*Example:*  
\( \sum_{n=1}^{\infty} \frac{2n}{n^2 + 1} \quad \lim_{n \to \infty} \frac{2n}{n^2 + 1} = 0 \) so, the series may converge OR diverge!

Using the integral test:

\[ \int_{1}^{\infty} \frac{2x}{x^2 + 1} \, dx = \lim_{b \to \infty} \left[ \int_{1}^{b} \frac{2x}{x^2 + 1} \, dx \right] \]

\[ = \lim_{b \to \infty} \left[ \ln(x^2 + 1) \right]_{1}^{b} = \infty - \ln(2) \quad \text{DIVERGES} \]
TEST: Comparison Test

If \( \sum_{n=1}^{\infty} u_n \) converges, and \( a_n \leq u_n \) then \( a_n \) converges.

Example: \( \sum_{n=1}^{\infty} \frac{1}{2 + n^3} \)

Since the integral test is difficult, we can try the comparison test. We'll choose the p-series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) because it is similar AND the terms will be greater than the terms in the main series.

In this p-series, \( p > 3 \), so it converges...

\[ \sum_{n=1}^{\infty} \frac{1}{2 + n^3} < \sum_{n=1}^{\infty} \frac{1}{n^3} \quad \text{CONVERGES} \]

If \( \sum_{n=1}^{\infty} u_n \) diverges, and \( a_n \geq u_n \) then \( a_n \) diverges.

Example: \( \sum_{n=1}^{\infty} \frac{2}{\sqrt[5]{5n} - 1} \)

If we use the comparison test, we can choose \( \sum_{n=1}^{\infty} \frac{2}{\sqrt[5]{5n}} \)

\[ \frac{2}{\sqrt[5]{5}} \sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}} \]

is p-series where \( p = 1/2 \) so, it diverges.

\[ \sum_{n=1}^{\infty} \frac{2}{\sqrt[5]{5n} - 1} > \sum_{n=1}^{\infty} \frac{2}{\sqrt[5]{5n}} \quad \text{DIVERGES} \]

(Note: the integral test could verify that this series diverges)

TEST: Limit Comparison Test

If \( \lim_{n \to \infty} \frac{a_n}{b_n} \) is a finite value (and non-zero), then

\[ \sum_{n=1}^{\infty} a_n \quad \text{AND} \quad \sum_{n=1}^{\infty} b_n \]

are either BOTH converging or BOTH diverging.

Example: \( \sum_{n=1}^{\infty} \frac{1}{2n + 1} \)

\[ \frac{1}{2n + 1} < \frac{1}{n} \quad \text{for all positive } n \]

\[ \sum_{n=1}^{\infty} \frac{1}{2n + 1} < \sum_{n=1}^{\infty} \frac{1}{n} \]

We know the harmonic series diverges, so the comparison test doesn't help...

\[ \lim_{n \to \infty} \frac{1}{2n + 1} = \lim_{n \to \infty} \frac{1}{2n + 1} \cdot \frac{n}{n} = \frac{1}{2} \]

There is a finite value, 1/2, and since \( \frac{1}{n} \) is diverging, then \( \frac{1}{2n + 1} \) must be diverging.
TEST: Ratio Test  
If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \) (if it exists), then \( \sum_{n=1}^{\infty} a_n \) converges if \( L < 1 \)  
diverges if \( L > 1 \)  
is inconclusive if \( L = 1 \)

Examples:
\[
\sum_{n=1}^{\infty} \frac{7^n}{(-3)^{n+1} \cdot n} = \frac{7}{9} + \frac{49}{54} + \frac{243}{243} + \ldots
\]

\[
\lim_{n \to \infty} \frac{7^{n+1}}{(-3)^{n+2} \cdot (n+1)} = \lim_{n \to \infty} \frac{7^{n+1}}{(-3)^{n+2} \cdot (n+1)} \cdot \frac{(-3)^n \cdot n}{(-3)^{n+1} \cdot n} = \frac{7}{3} \cdot \lim_{n \to \infty} \frac{n}{n+1} = \frac{7}{3}
\]
Since the limit \( L \) of the sequence > 1, the series DIVERGES.

\[
\sum_{n=1}^{\infty} \frac{3^n}{n!} = 3 + \frac{9}{2} + \frac{27}{6} + \ldots
\]

\[
\lim_{n \to \infty} \frac{3^{n+1}}{(n+1)!} = \lim_{n \to \infty} \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \lim_{n \to \infty} \frac{n!}{n+1} = 0
\]
Since the limit \( L \) of the sequence < 1, the series CONVERGES.

TEST: Nth Root Test  
If \( \lim_{n \to \infty} \sqrt[n]{|a_n|} = L \), then \( \sum_{n=1}^{\infty} a_n \) converges if \( L < 1 \)  
diverges if \( L > 1 \)  
is inconclusive if \( L = 1 \)

Examples:
\[
\sum_{n=0}^{\infty} \frac{2^n \cdot 3^n}{16^n}
\]

Using the nth root test, \( \lim_{n \to \infty} \sqrt[n]{\frac{2^n \cdot 3^n}{10^n}} = \lim_{n \to \infty} \frac{2^1 \cdot 3^1}{10^1} = \frac{18}{10} \)  
Since \( L = 9/5 > 0 \)  
the series DIVERGES.

\[
\sum_{n=0}^{\infty} \frac{2^n \cdot 3^n}{5^n}
\]

\[
L = \lim_{n \to \infty} \frac{\sqrt[n]{2^n \cdot 3^n}}{5^n} = \lim_{n \to \infty} \frac{\sqrt[n]{2} \cdot \sqrt[n]{3}}{\sqrt[n]{5}} = \lim_{n \to \infty} \frac{2 \cdot 1}{\sqrt[n]{5}} = \frac{2 \cdot 1}{\sqrt[n]{5}} = \frac{2}{5}
\]
Since \( L = 2/5 < 0 \)  
the series CONVERGES.
TEST: Alternating Series Test

An alternating series converges if \( \lim_{n \to \infty} a_n = 0 \)

AND \( 0 < a_{n+1} < a_n \) for all \( n \geq 1 \)

**Examples:**

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(2n)} = -\frac{1}{\ln(2)} + \frac{2}{\ln(4)} - \frac{3}{\ln(6)} + \ldots
\]

Using L'Hôpital's Rule

\[
\lim_{n \to \infty} \frac{1}{2n} = \lim_{n \to \infty} \frac{n}{\ln(2n)} = \frac{\infty}{\infty} = \text{inconclusive, so we'll use L'Hôpital's Rule}
\]

\[
= \lim_{n \to \infty} \frac{1}{2n} \left( \frac{1}{\ln(3)} \right) = 0
\]

So, the sequence converges and the series MAY converge...

check \( 0 < a_{n+1} < a_n \)

\[
0 < \frac{n+1}{3^n} < \frac{n}{(3)^{n-1}} \quad \text{"cross-multiply"}
\]

this is satisfied if \( (n + 1)(3)^{n-1} < n3^n \) \quad \text{"divide by } (n + 1)"

\[
\frac{n-1}{3} < \frac{n}{(n + 1)} \quad \text{"divide by } 3^n \quad \text{"}
\]

Since this is satisfied for \( n \geq 1 \), the series CONVERGES
Example: Find the Taylor (polynomial) series of the 4th order for the function $f(x) = \cos(2x)$

$$f(x) = \cos(2x) \quad f(0) = 1$$
$$f'(x) = -2\sin(2x) \quad f'(0) = 0$$
$$f''(x) = -4\cos(2x) \quad f''(0) = -4$$
$$f'''(x) = 8\sin(2x) \quad f'''(0) = 0$$
$$f^{(4)}(x) = 16\cos(2x) \quad f^{(4)}(0) = 16$$

$$f(x) = \sum_{n=0}^{4} \frac{f^{(n)}(0)}{n!} (x-a)^n$$

is the series of the 4th order...

$$= \frac{-1}{0!}(x)^0 + \frac{0}{1!}(x)^1 + \frac{-4}{2!}(x)^2 + \frac{0}{3!}(x)^3 + \frac{16}{4!}(x)^4$$

$$= 1 - 2x^2 + \frac{2}{3}x^4$$

Note the similarity of the graphs!

$$f(x) = \sum_{n=0}^{6} \frac{f^{(n)}(0)}{n!} (x-a)^n$$

is the series of the 6th order...

$$= \frac{-1}{0!}(x)^0 + \frac{0}{1!}(x)^1 + \frac{-4}{2!}(x)^2 + \frac{0}{3!}(x)^3 + \frac{16}{4!}(x)^4 + \frac{0}{5!}(x)^5 + \frac{-64}{6!}(x)^6$$

$$= 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}(x)^6$$

$$f(x) = \sum_{n=0}^{8} \frac{f^{(n)}(0)}{n!} (x-a)^n$$

is the series of the 8th order...

$$= 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}(x)^6 + \frac{256}{8!}(x)^8$$

NOTE: This is a MacLaurin Series, a special version of the Taylor Series. It occurs when $a = 0$

"A Taylor series about $x = 0$" is a MacLaurin series for $f(x)"
Example: Find the 5th non-zero terms in the Taylor Series generated by \( f(x) = \sqrt{x+1} \) at \( x = 0 \)

\[
f(x) = (x+1) - \frac{1}{2} \frac{1}{(x+1)^2} - \frac{1}{4} \frac{1}{(x+1)^3} - \frac{3}{8} \frac{1}{(x+1)^4} - \frac{7}{16} \frac{1}{(x+1)^5}
\]

\[
f(0) = 1 \quad f'(0) = \frac{1}{2} \quad f''(0) = -\frac{1}{4} \quad f'''(0) = -\frac{3}{8} \quad f''''(0) = -\frac{15}{16}
\]

Applying the formula:

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (x-0)^n = \frac{1}{6!} (x-0)^0 + \frac{1}{2!} (x-0)^1 + \frac{-1}{4!} (x-0)^2 + \frac{3}{8!} (x-0)^3 + \frac{-15}{4!} (x-0)^4 + ...
\]

First 5 terms...

\[
1 + \frac{1}{2} x - \frac{1}{8} x^2 + \frac{1}{16} x^3 - \frac{5}{128} x^4
\]
1) Find the Maclaurin Series of the 5th order for the function \( f(x) = \sin(2x) \)

2) Find the polynomial of order 4 at 0 for \( f(x) = e^{-x} \)
   Use this to approximate \( e^{0.5} \)

3) What is the coefficient of \((x - 2)^3\) in the Taylor Series generated by \( \ln(x) \) at \( x = 2 \)
4) \[ \sum_{n=1}^{\infty} \frac{(n+3)!}{3! \cdot n! \cdot 3^n} \]

Does the series converge or diverge?

5) \[ 1 + \frac{1}{\sqrt[5]{2}} + \frac{1}{\sqrt[5]{3}} + \frac{1}{\sqrt[5]{4}} + \ldots \]

Does the series converge or diverge?

6) \[ \sum_{n=1}^{\infty} \frac{3}{n(n+3)} + \frac{1}{7^n} = \]
1) Find the MacLaurin Series of the 5th order for the function $f(x) = \sin(2x)$

Series, Convergence, and Divergence

$$f(x) = \sin(2x)$$

$$f(0) = 0$$

$$f(x) = 2\cos(2x)$$

$$f(0) = 2$$

$$f''(x) = -4\sin(2x)$$

$$f''(0) = 0$$

$$f'''(x) = -8\cos(2x)$$

$$f'''(0) = -8$$

$$f^{(4)}(x) = 16\sin(2x)$$

$$f^{(4)}(0) = 0$$

$$f^{(5)}(x) = 32\cos(2x)$$

$$f^{(5)}(0) = 32$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

SOLUTIONS

Since a MacLaurin series is around $x = 0$, we'll let $a = 0$

$$f(x) \rightarrow 0! \cdot 0^0 + \frac{2}{1!} \cdot 0^1 + \frac{0}{2!} \cdot 0^2 + \frac{-8}{3!} \cdot 0^3 + \frac{0}{4!} \cdot 0^4 + \frac{32}{5!} \cdot 0^5$$

$$f(x) = 2x - \frac{4}{3}x^3 + \frac{1}{15}x^5$$

2) Find the polynomial of order 4 at 0 for $f(x) = e^{-x}$

Use this to approximate $e^{(.5)}$

$$f(x) = e^{-x}$$

$$f(0) = 1$$

$$f'(x) = -e^{-x}$$

$$f'(0) = -1$$

$$f''(x) = e^{-x}$$

$$f''(0) = 1$$

$$f'''(x) = -e^{-x}$$

$$f'''(0) = -1$$

$$f^{(4)}(x) = e^{-x}$$

$$f^{(4)}(0) = 1$$

$$e^{-x} = 1 + (-1) \frac{x}{1!} + (1) \frac{x^2}{2!} + (-1) \frac{x^3}{3!} + (1) \frac{x^4}{4!} + ...$$

$$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}$$

To approximate $e^{(.5)}$ we'll let $x = -1/2$

$$f(-1/2) = e^{-1/2}$$

$$f(-1/2) = 1 - (-1/2) + \frac{(-1/2)^2}{2} - \frac{(-1/2)^3}{6} + \frac{(-1/2)^4}{24}$$

$$\approx 1.64872$$

(approx)

$$1 + \frac{1}{2} + \frac{1}{8} \cdot + 1/48 + 1/384$$

$$= 1.64844$$

3) What is the coefficient of $(x - 2)^3$ in the Taylor Series generated by $\ln(x)$ at $x = 2$?

$$f(x) = \ln(x)$$

$$f(2) = \ln(2)$$

$$f'(x) = \frac{1}{x}$$

$$f'(2) = 1/2$$

$$f''(x) = \frac{-1}{x^2}$$

$$f''(2) = -1/4$$

$$f'''(x) = \frac{2}{x^3}$$

$$f'''(2) = 2/8$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$$\ln(2)(x-2) + \frac{1/2}{1!} (x-2) + \frac{-1/4}{2!} (x-2)^2 + \frac{2/8}{3!} (x-2)^3$$

Coefficient is $1/24$
4) \[ \sum_{n=1}^{\infty} \frac{(n + 3)!}{3! n! 3^n} \] Does the series converge or diverge?

Try the ratio test...

\[
\lim_{n \to \infty} \frac{\frac{(n + 4)!}{3! (n + 1)! 3^{n + 1}}}{\frac{(n + 3)!}{3! n! 3^n}} = \lim_{n \to \infty} \frac{(n + 4)!}{3! (n + 1)! 3^{n + 1} \cdot \frac{3! n! 3^n}{(n + 3)!}} = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9} \cdot 1.
\]

Since the limit < 1, the series CONVERGES.

5) \[ 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots \] Does the series converge or diverge?

Rewrite... \[ \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots \]

This is a p-series where \( p = 2/5 \)

Since \( p = 2/5 < 1 \), this series DIVERGES.

6) \[ \sum_{n=1}^{\infty} \frac{3}{n(n + 3)} + \frac{1}{7^n} = \]

\[
\sum_{n=1}^{\infty} \frac{3}{n(n + 3)} + \sum_{n=1}^{\infty} \left( \frac{1}{7} \right)^n \quad \text{geometric series}
\]

\[ \Rightarrow \sum_{n=1}^{\infty} \left( \frac{1}{7} \right)^n = \frac{\frac{1}{7}}{1 - \frac{1}{7}} = \frac{1}{6} \]

So, \[ \sum_{n=1}^{\infty} \left( \frac{1}{7} \right)^n = \frac{1}{6} \]

using partial fractions...

\[ \frac{3}{n(n + 3)} = \frac{A}{n} + \frac{B}{n + 3} \]

\[ \frac{3}{n(n + 3)} = \frac{A(n + 3) + Bn}{n(n + 3)} \]

\[ A = 1 \]

\[ B = -1 \]

The sum is \[ \frac{1}{6} + \frac{11}{6} = 2 \]
Improper Integrals

At the 2015 Annual Math Dinner/Formal...

"Those two always diverge from acceptable behavior..."

"Why is everyone staring at us? Are we underdressed?"

"No, dummy. It's because you used the wrong fork at dinner!"

Examples→
Improper Integrals

Definition: A definite integral where the integrand has a discontinuity between the bounds of integration.
(or, the upper/lower bound is \( +/- \infty \))

An improper integral can be evaluated using limits!

if the limit exists (and is finite), it converges

if the limit does not exist (or, is infinite), it diverges

Example:
\[
\int_{1}^{\infty} \frac{1}{x^{1.1}} \, dx
\]

Step 1: If possible, sketch a graph

We're looking for the area under the curve.
(Since it goes on forever, we are looking for the value of convergence it approaches.)

Step 2: Evaluate the integral, substituting limits

\[
\int_{1}^{\infty} x^{-1.1} \, dx = \left[ \frac{x^{-0.1}}{-1} \right]_{1}^{\infty} = \lim_{b \to \infty} \frac{1}{-1b^{-0.1}} - \frac{1}{-11^{-0.1}}
\]

("bottom heavy", so it goes to 0)

\[
= 0 - (-10) = 10
\]

Example:
\[
\int_{0}^{\ln 4} \frac{1}{x^2 e^x} \, dx
\]

\[
\int_{0}^{\ln 4} -1 x^{-2} e^x \, dx = -1 \int_{0}^{\ln 4} \frac{1}{x^2} \, dx = -1 \left[ -\frac{1}{x} \right]_{0}^{\ln 4} = -e^{-\ln 4} - e^{-0} = \infty
\]

Since 1/0 is undefined, this integral diverges

Since the derivative of \( \frac{1}{x} \) is \( -x^{-2} \),
we insert a -1
Comparison Test: Determining Convergence/Divergence

“When it’s difficult to evaluate an integral, try a similar equation.”

Example: Does \( \int_1^\infty \frac{dx}{1 + e^x} \) converge or diverge?

\( \frac{1}{1 + e^x} \) is difficult to integrate...

However, \( \frac{1}{e^x} \) is much easier....

\[
\int_1^\infty \frac{1}{e^x} \, dx = \int_1^\infty e^{-x} \, dx = \left[ -e^{-x} \right]_1^\infty = \lim_{b \to \infty} -e^{-b} - (-e^{-1}) = 0 + \frac{1}{e^1} = \frac{1}{e^1}
\]

Since the larger value (greater area) converges, the lesser value must converge, too...

Example: Does \( \int_{-\infty}^\infty \frac{2 + \cos \Theta}{\Theta} \, d\Theta \) converge or diverge?

Again, this integral is difficult to find.

But, \( \frac{2}{\Theta} \) is similar and much easier.

\[
\frac{2 + \cos \Theta}{\Theta} > \frac{2}{\Theta}
\]

Since the smaller value diverges, the larger value must diverge, too.
Determining Convergence/Divergence: Comparison Test

Example: Does \( \int_{1}^{\infty} \frac{e^{-x}}{\sqrt[4]{x}} \, dx \) converge or diverge?

First, let's rewrite the equation:

\[
\frac{1}{\sqrt{x}} > \frac{1}{e^x \sqrt{x}}
\]

for all \( x \geq 1 \)

\[
\int_{1}^{\infty} \frac{1}{\sqrt{x}} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{(x)^{\frac{1}{2}}} \, dx \quad \rightarrow \quad \lim_{b \to \infty} \left[ \frac{2}{(x)^{\frac{1}{2}}} \right]_{1}^{b} = \infty - 2
\]

DIVERGES

Since the 'larger' equation diverges, the comparison test is inconclusive....

Now, let's test another function....

\[
\frac{1}{e^x} > \frac{1}{e^x \sqrt{x}}
\]

for all \( x \geq 1 \)

\[
\int_{1}^{\infty} \frac{1}{e^x} \, dx = \lim_{b \to \infty} \int_{1}^{b} e^{-x} \, dx \quad \rightarrow \quad \lim_{b \to \infty} \left[ e^{-x} \right]_{1}^{b} = \lim_{b \to \infty} \left( \frac{-1}{e^x} \right)_{1}^{b} = -1 + \frac{1}{e}
\]

CONVERGES

Since the 'larger' equation converges, the integral must converge, too!
Using Inverse Trigonometry Function

What is the area under the curve \( y = \frac{1}{x^2 + 1} \) in Quadrant I?

Step 1: If possible, sketch the graph

The curve approaches 0 in both directions.

Step 2: Determine boundaries of integrand (ends of the integral)

We're looking for the area in quadrant I. (under the curve and above the x-axis)

Since the curve never gets to the x-axis, the boundaries of the integral will be

\( x = 0 \) and \( \infty \)

Step 3: Evaluate integral

\[
\int_{0}^{\infty} \frac{1}{x^2 + 1} \, dx = \lim_{b \to \infty} \int_{0}^{b} \frac{1}{x^2 + 1} \, dx = \lim_{b \to \infty} \tan^{-1}(x) \bigg|_{0}^{b} = \frac{\pi}{2} - 0 = \frac{\pi}{2}
\]

\( \tan\left(\frac{\pi}{2}\right) \) is undefined

\( \tan(0) = 0 \)

Evaluate \[
\int_{1}^{\infty} \frac{\tan^{-1}(t)}{1 + t^2} \, dt
\]

\[
\int_{1}^{\infty} \tan^{-1}(t) \frac{1}{1 + t^2} \, dt = \lim_{b \to \infty} \left( \tan^{-1}(t) \right)^2 \bigg|_{1}^{b} = \lim_{b \to \infty} \frac{(\tan^{-1}(b))^2}{2} - \frac{(\tan^{-1}(1))^2}{2}
\]

\[
\frac{\left(\frac{\pi}{2}\right)^2}{2} - \frac{\left(\frac{\pi}{4}\right)^2}{2} = \frac{\pi^2}{8} - \frac{\pi^2}{32}
\]

\[
\frac{3\pi^2}{32} \approx .925
\]
Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers

Mathplane Express for mobile at Mathplane.ORG

Also, content at the Mathplane stores, available at TES and TeachersPayTeachers. And, Facebook, Google +, and Pinterest
Twelve hours later, the Kodak family did try one more pose...
(The evening photo wasn’t much better....)