## Geometry: Proofs and Postulates

### Definitions, Notes, & Examples

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<tr>
<th>Statements</th>
<th>Reasons</th>
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<tr>
<td>1. $\overline{AD}$ and $\overline{BC}$ bisect each other</td>
<td>1. Given</td>
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<tr>
<td>2. $\overline{AM} \cong \overline{DM}$; $\overline{CM} \cong \overline{BM}$</td>
<td>2. Definition of bisector</td>
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<tr>
<td>3. $\angle AMC \cong \angle BMD$</td>
<td>3. Vertical angles are congruent</td>
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<tr>
<td>4. $\triangle AMC \cong \triangle DMB$</td>
<td>4. Side-Angle-Side (SAS)</td>
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<tr>
<td>5. $\overline{AC} \cong \overline{BD}$</td>
<td>5. CPCTC (Corresponding Parts of Congruent Triangles are Congruent)</td>
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</table>

Topics include triangle characteristics, quadrilaterals, circles, midpoints, SAS, and more.

Mathplane.com
Proofs and Postulates: Triangles and Angles

Postulate: A statement accepted as true without proof.

I. A Straight Angle is $180^\circ$

II. Supplementary Angles add up to $180^\circ$

$m\angle A + m\angle B = 180^\circ$

Example:

$\angle xyr$ and $\angle yrz$ are supplementary angles.

And, although they are not adjacent, $\angle S$ and $\angle xyr$ are supplementary as well.

Theorem: A statement or assertion that can be proven using rules of logic.

III. Vertical Angles are congruent

$\angle R \cong \angle S \quad \angle X \cong \angle Y$

Examples:

Using postulates and math properties, we construct a sequence of logical steps to prove a theorem.
Parallel Line Postulate: If 2 parallel lines are cut by a transversal, then their corresponding angles are congruent.

![Diagram of parallel lines and transversal]

\[ \angle 1 \cong \angle 2 \]

IV. If parallel lines are cut by a transversal, the alternate interior angles are congruent.

![Diagram of alternate interior angles]

A simple sketch can show the parallel line postulate.

Note: Moving each point the same distance and direction will produce a parallel line (and a corresponding angle).

Proof of parallel lines/alt. interior angles:

<table>
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<tr>
<td>1. ( l \parallel m )</td>
<td>1. given</td>
</tr>
<tr>
<td>2. ( t ) is transversal</td>
<td>2. given (def. of transversal)</td>
</tr>
<tr>
<td>3. ( \angle D \cong \angle E )</td>
<td>3. If parallel lines cut by transversal, then corresponding angles are congruent</td>
</tr>
<tr>
<td>4. ( \angle C \cong \angle D )</td>
<td>4. Vertical angles congruent</td>
</tr>
<tr>
<td>5. ( \angle C \cong \angle E )</td>
<td>5. Substitution</td>
</tr>
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</table>

Examples:

If \( \angle 2 = 70^\circ \) and \( r \) is parallel to \( s \),

1. \( 4 = 110^\circ \) (2 and 4 are supplementary)
2. \( 3 = 70^\circ \) (3 and 2 are corresponding)
3. \( 5 = 70^\circ \) (3 and 5 are vertical angles)
4. \( 6 = 70^\circ \) (3 and 6 are alt. interior angles)
5. \( 1 = ? \) (\( p \) is not parallel to \( r \) or \( s \))
V. The sum of the interior angles of a triangle is $180^\circ$ (Theorem)

$$m \angle A + m \angle B + m \angle C = 180^\circ$$

Examples:

\[
\begin{align*}
x + 43 + 85 &= 180 \text{ degrees} \\
x &= 52 \text{ degrees}
\end{align*}
\]

\[
\begin{align*}
S + 40 + 80 &= 180 \\
S &= 60 \text{ degrees}
\end{align*}
\]

\[
\begin{align*}
T + S &= 180 \text{ degrees} \\
T + 60 &= 180 \\
So, \ T &= 120 \text{ degrees}
\end{align*}
\]

** Illustrates the triangle (remote) exterior angle theorem: the measure of an exterior angle equals the sum of the 2 non-adjacent interior angles.

Informal Proof: $1 + 2 + 3 = 180^\circ$

Add parallel line to one of the sides

\[
\begin{align*}
A + 1 + B &= 180 \text{ degrees (straight angle and addition postulate)} \\
A &= 2 \text{ and } B = 3 \text{ (parallel lines cut by transversal, then alt. interior angles are congruent)} \\
2 + 1 + 3 &= 180 \text{ degrees (substitution)}
\end{align*}
\]
Tools to consider in Geometry proofs:

1) Using CPCTC (Corresponding Parts of Congruent Triangles are Congruent) after showing triangles within the shapes are congruent.
   Try
   a) reflexive property
   b) vertical angles are congruent
   c) alternate interior angles (formed by parallel lines cut by a transversal) are congruent
      Then, verify congruent triangles by SAS, SSS, ASA, AAS, HL.

2) Common properties and theorems
   a) Triangles are 180°; Quadrilaterals are 360°
   b) Opposite sides of congruent angles are congruent (isosceles triangle)
   c) Perpendicular bisector Theorem
      (All points on perpendicular bisector are equidistant to endpoints)

3) Other geometry basics:
   a) All radii of a circle are congruent
   b) Supplementary angles (180°); Complementary angles (90°)
   c) Midpoints and medians divide segments into congruent parts

   d) Altitudes form right angles
"Definition of Supplementary"

If 2 angles are supplementary to the same angle, then the angles are congruent.

Angle 3 is supplementary to angle 1...
Angle 2 is supplementary to angle 1...
Therefore, angles 3 and 2 are congruent!

If 2 angles are supplementary to congruent angles, then the angles are congruent.

Angle 2 is supplementary to angle 1
Angle 3 is supplementary to angle 4
Since angles 1 and 4 are congruent, then therefore, angles 2 and 3 are congruent!

Given: \( \angle 1 \cong \angle 2 \)
BE \( \cong BD \)

Prove: \( \triangle ABE \cong \triangle CBD \)

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<td>1) ( BE \cong BD )</td>
<td>1) Given</td>
</tr>
<tr>
<td>2) ( \angle 1 \cong \angle 2 )</td>
<td>2) Given</td>
</tr>
<tr>
<td>3) ( \angle CDB ) is supp to 1</td>
<td>3) Definition of Supplementary</td>
</tr>
<tr>
<td>4) ( \angle AEB ) is supp to 2</td>
<td>4) Def. of Supplementary (diagram)</td>
</tr>
<tr>
<td>5) ( \angle CDB \cong \angle AEB )</td>
<td>5) If angles supp. to congruent angles, then angles congruent</td>
</tr>
<tr>
<td>6) ( \angle CBD \cong \angle ABE )</td>
<td>6) Vertical angles congruent</td>
</tr>
<tr>
<td>7) ( \angle ABE \cong \angle CBD )</td>
<td>7) Angle-Side-Angle (6, 1, 5)</td>
</tr>
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</table>
Explain the implication of each hypothesis...

1) If \( \overline{AD} \) bisects \( \overline{EC} \), then _________________

2) If \( \overline{AC} \) bisects \( \angle BAD \), then _________________

3) If \( \overrightarrow{LM} \parallel \overrightarrow{PQ} \), then _________________

4) If lines \( LM \) and \( NP \) intersect at point \( Q \), then _________________

5) If \( O \) is the center of the circle, then _________________

6) If \( \overline{AC} \parallel \overline{BD} \), then _________________

7) \( \angle MNP \) and \( \angle MNQ \) are complementary angles, then _________________

8) If \( \angle RGT \) and \( \angle HGM \) are right angles, then _________________
1) If $\overline{AD}$ bisects $\overline{BC}$, then $\overline{BC} = \overline{BE}$

2) If $\overline{AC}$ bisects $\angle BAD$, then angles $\overline{BAC}$ and $\overline{DAC}$ are congruent

NOTE: It does not imply that $\overline{BC} = \overline{CD}$. It might be... Or, they may not be...

3) If $\overline{LM} \parallel \overline{PQ}$, then $\overline{LMQ}$ and $\overline{LMP}$ are right angles

4) If lines $\overline{LM}$ and $\overline{NP}$ intersect at point $Q$, then angles $\overline{NQM}$ and $\overline{LQP}$ are congruent (by vertical angles)

5) If $O$ is the center of the circle, then $\overline{OA} \cong \overline{OB}$ (because all radii are congruent)

6) If $\overline{AC} \parallel \overline{BD}$, then $\overline{AB} \cong \overline{CD}$ (by subtraction of $\overline{BC}$)

7) $\angle MNP$ and $\angle MNQ$ are complementary angles, then $\overline{MNQ}$ is a right angle (and $\overline{MN} \perp \overline{NQ}$)

8) If $\angle RGT$ and $\angle HGM$ are right angles, then $\angle RGH \cong \angle TMG$ (by subtraction OR by using complementary angles and transitive)

(by subtraction OR by using complementary angles and transitive)
... Stranded somewhere in the (Bermuda) Triangle...

SAS is not a distress signal

When you're in the Bermuda Triangle, SOS is more useful than SAS!!

Examples -→
Proving a Median of a Triangle: Example

Given: \( \overline{CM} \) bisects \( \angle BCD \)
\[ \overline{DC} \cong \overline{BC} \]
Prove: \( \overline{AM} \) is a median of \( \triangle BDA \)

Step 1: "Label the picture."
\[ \overline{CM} \text{ bisects } \angle BCD \]
\[ \overline{DC} \cong \overline{BC} \]

What are we trying to prove? \( \overline{DM} \cong \overline{MB} \)
(definition of a median)

Step 2: Determine a Strategy
To prove a median, I need to show a segment bisects the opposite side. (def. of a median).

Notice triangles \( \triangle CMD \) & \( \triangle CMB \). They include \( \overline{DM} \) & \( \overline{MB} \).
If I can show \( \triangle CMD = \triangle CMB \), then I can use CPCTC to prove that \( \overline{DM} = \overline{MB} \).

Step 3: Write the Proof (describing your approach and strategy)

<table>
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<tbody>
<tr>
<td>1) ( \overline{DC} = \overline{BC} ) ( \overline{CM} \text{ bisects } \angle BCD )</td>
<td>1) Given</td>
</tr>
<tr>
<td>2) ( \angle DCM = \angle BCM )</td>
<td>2) Definition of Angle Bisector</td>
</tr>
<tr>
<td>3) ( \overline{MC} = \overline{MC} )</td>
<td>3) Reflexive Property</td>
</tr>
<tr>
<td>4) ( \triangle DCM = \triangle BCM )</td>
<td>4) Side-Angle-Side (SAS) postulate</td>
</tr>
<tr>
<td>5) ( \overline{DM} = \overline{BM} )</td>
<td>5) Corresponding Parts of Congruent Triangles are Congruent (CPCTC)</td>
</tr>
<tr>
<td>6) ( \overline{AM} ) is median of ( \triangle ABD )</td>
<td>6) Definition of a Median (A line segment joining vertex of triangle to midpoint of opposite side)</td>
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</table>
Proof Example

"Prove the base angles of an isosceles triangle are congruent."

Given: Isosceles Triangle
Prove: Base Angles are congruent

Step 1: Draw pictures and label.

What is given? A triangle with 2 sides of equal length.
(Definition of an Isosceles Triangle)

What are we trying to prove? \( \angle B \cong \angle C \)

Step 2: Determine a strategy.
Try dividing into triangles.
Use properties of congruent triangles.
Then, CPCTC.

Step 3: Write the proof (describing your strategy!)

<table>
<thead>
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<tbody>
<tr>
<td>1. ( AB \cong AC )</td>
<td>1. Given/Definition of Isosceles Triangle</td>
</tr>
<tr>
<td>2. Draw a Median</td>
<td>2. An angle has one median</td>
</tr>
</tbody>
</table>
| 3. \( BM \cong CM \) | 3. Definition of a median
(A line segment joining a vertex and
the midpoint of the opposite side) |
| 4. \( AM \cong AM \) | 4. Reflexive Axiom |
| 5. \( \triangle AMB = \triangle AMC \) | 5. SSS congruence postulate
(If 3 sides of one triangle are congruent to
the corresponding sides of another triangle,
then the triangles are congruent) |
| 6. \( \angle B \cong \angle C \) | 6. CPCTC (Corresponding Parts of Congruent Triangles are Congruent) |

Alternate Proof: (Using angle bisector and Side-Angle-Side)

<table>
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<tbody>
<tr>
<td>1. ( \triangle ABC ; \ AB \cong AC )</td>
<td>1. Given; Def. of Isosceles triangle</td>
</tr>
<tr>
<td>2. ( AM ) is an angle bisector</td>
<td>2. An angle has one bisector</td>
</tr>
<tr>
<td>3. ( \angle BAM \cong \angle CAM )</td>
<td>3. Def. of Angle Bisector</td>
</tr>
<tr>
<td>4. ( AM \cong AM )</td>
<td>4. Reflexive Property</td>
</tr>
<tr>
<td>5. ( \triangle BAM = \triangle CAM )</td>
<td>5. SAS postulate</td>
</tr>
<tr>
<td>6. ( \angle B = \angle C )</td>
<td>6. CPCTC</td>
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</table>
**Two Basic Circles Proofs**

NOTE: Both proofs use SAS (Side-Angle-Side)...

However, the first proof utilizes the midpoints to get congruent segments and, the second proof uses ‘all radii congruent’ to get congruent sides...

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**Statements**

1. M is midpoint of BC
   2. BM = CM
   3. ∠AMB = ∠CMD
   4. ∆AMB = ∆DMC

**Reasons**

1. Given
2. Definition of midpoint
3. Vertical angles congruent
4. Side-angle-side

---

**Statements**

1. Circle P
2. TP = IP = SP = RP
3. ∠TPS = ∠RPI
4. ∆TSP = ∆RIP

**Reasons**

1. Given
2. All radii are congruent
3. Vertical angles congruent
4. Side-angle-side

---

**Applying “subtraction” or “complementary angles”**

**Given:** \( AR \perp RC \)

**Prove:** \( \angle 1 \cong \angle 3 \)

**Columns:**

**Statements**

1. \( AR \perp RC \)
2. \( BR \perp RD \)
3. \( \angle ARC \) and \( \angle BRD \) are right angles
4. \( \angle 1 \) and \( \angle 2 \) are complementary angles
5. \( \angle 3 \) and \( \angle 2 \) are complementary angles

**Reasons**

1. Given
2. Given
3. Definition of Perpendicular (Perpendicular lines form right angles)
4. Complementary angles for right angles
5. Complementary angles for right angles
6. If 2 angles are complementary to congruent angles, then the 2 angles are congruent (also, transitive property)

---

**Columns:**

**Statements**

1. \( AR \perp RC \)
2. \( BR \perp RD \)
3. \( \angle ARC \) and \( \angle BRD \) are right angles
4. If 2 angles are complementary to congruent angles, then the 2 angles are congruent (also, transitive property)

**Reasons**

1. Given
2. Given
3. Definition of Perpendicular (Perpendicular lines form right angles)
4. All right angles are congruent
5. Reflexive property
6. Subtraction Property (If congruent angles are subtracted from congruent angles, then the difference are congruent)
EXAMPLE: Prove diagonals of a rectangle are congruent and bisect each other.

Given: Rectangle ABDC
Prove: BC = AD and BC, AD bisect each other

PROOF

1) AC = BD
   AB = CD
   Definition of Rectangle
   (opposite sides are congruent)
   (all angles are congruent, right angles)
   AB || CD
   AC || BD

2) ∠CBD ≅ ∠BCA
   ∠CAD ≅ ∠BDA
   Parallel lines cut by a transversal, then alternate interior angles are congruent

3) △ ACM = △ DBM
   Congruent triangles
   Angle-Side-Angle

4) AM = DM
   CM = BM
   CPCTC

5) BC and AD bisect each other
   Definition of Bisector
   (A line, ray, or segment that cuts a segment into 2 congruent parts)

6) CD = CD
   Reflexive Property

7) ∠C = ∠D = 90°
   AC = BD
   Definition of Rectangle

8) △ ACD = △ BDC
   Congruent triangles
   Side-Angle-Side

9) BC = AD
   CPCTC

Note: Pythagorean theorem can show that diagonals are equal
Proof Example
Prove the "Parallelogram Diagonals Theorem"
(The Diagonals of a Parallelogram Bisect each other)

Given: Parallelogram
Prove: Diagonals Bisect Each Other

(Draw Picture)

(Establish Strategy)
Diagonals bisecting each other implies that congruent line segments are inside the parallelogram.
(Also, notice that diagonals create triangles.)

(Label "things you know" about the Given (Parallelogram))

PROOF

1) \( AB \cong DC \)
   \( AD \cong BC \)
   \( \overline{AD} \parallel \overline{BC} \)
   \( \overline{AB} \parallel \overline{DC} \)

   Definition of a Parallelogram
   (opposite sides are congruent)
   (opposite sides are parallel)

2) \( \angle ACD \cong \angle CAB \)
   \( \angle ABD \cong \angle CDB \)

   If two parallel lines are cut by a transversal, then alternate interior angles are congruent.

3) \( \triangle ABE \cong \triangle CDE \)

   ASA (Angle-Side-Angle)
   Triangles are congruent

4) \( \overline{BE} \cong \overline{DE} \)
   \( \overline{AE} \cong \overline{CE} \)

   CPCTC
   (Corresponding Parts of Congruent Triangles are Congruent)

5) \( \overline{AC} \) bisects \( \overline{BD} \)
   and \( \overline{BD} \) bisects \( \overline{AC} \)

   Definition of a Bisector
   (A line, ray, or segment that cuts a segment into 2 congruent parts)

("Diagonals of a Parallelogram bisect each other")
Prove that the **shortest** distance between a point and a line is a perpendicular line segment.

Step 1: Write out the Given and Prove statements

**Given:** Line AB with external point X
- Line segment XY is perpendicular to AB
- Segment XC is non-perpendicular to AB

**Prove:** Segment XY is shorter than segment XC

Step 2: Draw a diagram to clarify

![Diagram showing points A, C, Y, and B with line AB and point X at an angle to AB, forming triangle XCY](image)

Step 3: Determine a strategy

We need to prove that XY < XC.
- We see that we're dealing with a right triangle (angles X and C are both acute, and Y is 90 degrees).
- And, we know that size of angles indicates size of opposite sides.

Step 4: Proof

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</table>
| 1. Line AB with external point X
  XY ⊥ AB | 1. Given |
| 2. ∠XYC is a right angle | 2. Perpendicular lines form right angle |
| 3. ∠XCY is an acute angle | 3. (Interior angles of Triangle add up to 180°)
  - Non-right angles of a right triangle (i.e. △XYC)
  - are always acute (< 90°) |
| 4. ∠XCY < ∠XYC | 4. Definition of Acute Angle
  (Angles that are less than 90°) |
| 5. XY is shorter than XC | 5. In a triangle, the side opposite the smaller angle is shorter than a side opposite a larger angle
  ∠XCY < ∠XYC so, XY < XC |
No-Choice Theorem Examples

Given: $\angle DBC \cong \angle E$
Prove: $\angle A \cong \angle BDC$

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<td>1. Given</td>
</tr>
<tr>
<td>2. $\angle C \cong \angle C$</td>
<td>2. Reflexive Property</td>
</tr>
<tr>
<td>3. $\angle A \cong \angle BDC$</td>
<td>3. No-Choice Theorem</td>
</tr>
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</table>

(If 2 angles of one triangle are congruent to 2 angles of another triangle, then the 3rd angles of both triangles are congruent)

Note: The angles are congruent.
So, the triangles are similar.
(We need at least one pair of congruent sides for congruent triangles)

Given: $\angle ABC \cong \angle ACD$
$\angle ACB \cong \angle D$

Are the triangles congruent?

Since two angles are congruent, the 3rd angles must be congruent (no-choice theorem)

We have angle-angle-angle...
(Similar Triangles)

BUT, the triangles may or may not be congruent...

NOTE: $\overline{AC}$ in $\triangle ABC$ does not correspond to $\overline{AC}$ in $\triangle ACD$
"Noah's Arc"

"Perhaps you misunderstood the command to 'build an ark'?

"I suppose I did..."

"Hey, at least it floats."

Eventually, Noah realizes that this assignment was NOT a geometry construction.
Thanks for visiting. (Hope it helped!)

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