Implicit Differentiation

Notes, examples, applications, and practice test (with solutions)

Topics include logarithms, inverse trig, tangent lines, graphing, related rates, and more.
Implicit Differentiation Notes and Examples

Explicit vs. Implicit Form:

Equations involving 2 variables are generally expressed in explicit form

\[ y = f(x) \]

In other words, one of the two variables is explicitly given in terms of the other.

Equations where relationships are not given explicitly are in implicit form.

\[ 2x - y = 4 \]
\[ xy = 1 \]
\[ x^3 + 2xy + y^2 = 0 \]

(explicit form: put in the input variable, and easily get the other)

(implicit form: the relationship between x and y isn’t easily seen)

Sometimes it is possible to change the form from implicit to explicit...

\[ 2x - y = 4 \implies y = 2x - 4 \]
\[ xy = 1 \implies y = \frac{1}{x} \]

…. But, other times it is very difficult or impossible to express in explicit form.

\[ x^2 + 2xy + y^2 = 0 \implies y = ? \]

So, to find the derivative, implicit differentiation is an easier approach.

Implicit Differentiation:

Method:
1) Take derivatives
2) When taking derivative of y, insert \( \frac{dy}{dx} \) (or \( y' \))
3) Solve for \( \frac{dy}{dx} \) (or \( y' \))

**Implicit Differentiation Example:**

\[ x^2 - 2y^3 + 4x = 2 \]
\[ 2x - 6y^2 \frac{dy}{dx} + 4 = 0 \]
\[ \frac{dy}{dx} = \frac{-2x - 4}{-6y^2} \]
\[ \frac{dy}{dx} = \frac{x + 2}{3y^2} \]

Example: Find the derivative with respect to x of

\[ x^2 + 2xy + y^2 = 0 \]
\[ 2x + 2(1y + 1y'x) + 2yy' = 0 \]
\[ 2x + 2y + 2xy' + 2yy' = 0 \]
\[ 2x + 2y = -2xy' - 2yy' \]
\[ x + y = y'(-x - y) \]
\[ y' = \frac{x + y}{-(x + y)} = -1 \]
Verifying Implicit Differentiation: An Example

Find the derivative of \( x^2 + y^2 = 25 \)

**Implicit Differentiation:**

\[ x^2 + y^2 = 25 \]
\[ 2x + 2yy' = 0 \]
\[ 2yy' = -2x \]
\[ y' = \frac{-x}{y} \]

Notice that implicit differentiation used fewer steps and easier equations!

**Graph:**

Graph of \( x^2 + y^2 = 25 \) with points: (0, -5), (5, 0), \((\sqrt{12.5}, \sqrt{12.5})\), and \((-\sqrt{12.5}, -\sqrt{12.5})\).

**Slope (rate of change) at:**

\( (0, -5): \quad \frac{-5}{0} = \text{undefined} \)

\( (5, 0): \quad \frac{5}{0} = \text{undefined} \)

\( (\sqrt{12.5}, \sqrt{12.5}): \quad \frac{-\sqrt{12.5}}{\sqrt{12.5}} = -1 \)

**Explicit Differentiation:**

\[ x^2 + y^2 = 25 \]

Change to implicit form:

\[ y^2 = 25 - x^2 \]
\[ y = \pm \sqrt{25 - x^2} \]

Find derivative (using power rule):

\[ y' = \frac{1}{2} (25 - x^2)^{-\frac{1}{2}} (-2x) \]

Simplify the result.

\[ y' = \frac{-x}{(25 - x^2)^{\frac{1}{2}}} \]

Then, for \( y = -\sqrt{25 - x^2} \)

\[ y' = (-1) \frac{1}{2} (25 - x^2)^{-\frac{1}{2}} (-2x) \]

\[ y' = \frac{x}{(25 - x^2)^{\frac{1}{2}}} = \frac{x}{-y} \quad \text{Same solution} \]
Example: Given \( x^2 + xy + y^2 = 3 \) Find \( \frac{dy}{dx} \)

\[
3x^2 + \left[ 1 \cdot y + x \cdot 1 \left( \frac{dy}{dx} \right) \right] + 2y \left( \frac{dy}{dx} \right) = 0
\]

\[
3x^2 + y + x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0
\]

\[
x \frac{dy}{dx} + 2y \frac{dy}{dx} = -3x^2 - y
\]

\[
\frac{dy}{dx} = \frac{-3x^2 - y}{x + 2y}
\]

Example: Given \( x^2 - 2x^2y + 3xy^2 = 38 \)

Evaluate the derivative at \((2,3)\)

Use implicit differentiation

\[
3x^2 - 2 \left[ 2xy + 1 \cdot y^2 \cdot x^2 \right] + 3 \left[ 1 \cdot y^2 + 2y \cdot y' \cdot x \right] = 0
\]

Simplify

\[
3x^2 - 4xy - 2y^2 x^2 + 3y^2 + 6y \cdot y' \cdot x = 0
\]

\[
6y \cdot y' \cdot x - 2y^2 x^2 = -3x^2 + 4xy - 3y^2
\]

\[
y' = \frac{-3x^2 + 4xy - 3y^2}{6xy - 2x^2}
\]

Plug in \((2,3)\)

\[
\frac{-3(4) + 4(6) - 3(9)}{6(6) - 2(4)} = \frac{-15}{28}
\]

\[
\int x^2 \, dx
\]

\[
\lim_{x \to \infty} \int f(x) \, dx
\]
Implicit Differentiation and Tangent Lines

Find the equation of the line tangent to \( x^2 + xy - y^2 = 1 \) at \((2, 3)\)

To find the equation of a line, we need the slope and a point.

The point is given: \((2, 3)\)

And, the slope is the *instantaneous rate of change (IROC)* at the given point.

Use implicit differentiation to find the IROC

\[
2x + (1)y + x(1) \frac{dy}{dx} - 2y \frac{dy}{dx} = 0
\]

then, the IROC at \((2, 3)\) is

\[
x \frac{dy}{dx} - 2y \frac{dy}{dx} = -2x - y \\
\frac{dy}{dx} = \frac{-2x - y}{x - 2y}
\]

So, the slope is \( \frac{7}{4} \)

\[
y - 3 = \frac{7}{4} (x - 2)
\]

\[
y = \frac{7}{4} x - \frac{1}{2}
\]

Find the equation of the line tangent to \( 2xy + \pi \sin y = 2 \pi \) at \((1, \frac{\pi}{2})\)

Use implicit differentiation to find the IROC, which is the slope of the tangent lines.

\[
2 \left[ (1)y + x(1) \frac{dy}{dx} \right] + \pi \left( \cos y \right) \frac{dy}{dx} = 0
\]

then, the slope is

\[
2y + 2x \frac{dy}{dx} + \pi \left( \cos y \right) \frac{dy}{dx} = 0
\]

\[
\frac{dy}{dx} = \frac{-2y}{2x + \pi \left( \cos y \right)}
\]

and, the equation of the tangent line is:

\[
y - \frac{\pi}{2} = -\frac{\pi}{2} (x - 1)
\]

\[
y = -\frac{\pi}{2} x + \pi
\]
Implicit differentiation to find slope

**Example:** Find slope of \( x^2 + y^2 = 13 \) at \((-2, 3)\) and at \((3, 2)\)

Find \( y' \):

\[
2x + 2yy' = 0
\]

\[
2yy' = -2x
\]

\[
y' = \frac{-2x}{2y}
\]

\[
y' = \frac{-x}{y}
\]

Then, to find slope at \((-2, 3)\):

\[
y' = \frac{-(-2)}{(3)} = \frac{2}{3}
\]

at \((3, 2)\):

\[
y' = \frac{-3}{2}
\]

**Example:** Find lines that are tangent and normal to \( x^2 y^2 = 9 \) at the point \((-1, 3)\)

Utilize implicit differentiation to find \( y' \)

(product rule)

\[
2xy^2 + 2yy'x^2 = 0
\]

(solve for \( y' \))

\[
2yy'x^2 = -2xy^2
\]

\[
y' = \frac{-2xy^2}{2yx^2} = \frac{-y}{x}
\]

Instantaneous rate of change \( \frac{dy}{dx} \) at \((-1, 3)\):

\[
\frac{-3}{-1} = 3
\]

Slope of tangent: 3 \hspace{1cm} Equation of tangent line: \( y - 3 = 3(x + 1) \)

\[
y = 3x + 6
\]

Slope of normal: -1/3 \hspace{1cm} Equation of normal line: \( y - 3 = -\frac{1}{3}(x + 1) \)

\[
y = \frac{1}{3}x + \frac{8}{3}
\]

\[
x + 3y = 8
\]
Example: Given the curve \( x^2 - xy + y^2 = 16 \)

Find the coordinate(s) where the tangents are vertical:

**SOLUTION:**

(If a tangent line is horizontal, then the slope is 0)

If a tangent line is vertical, then the slope is undefined!

To find the instantaneous rate of change, find the derivative:

(implicit differentiation)

\[
2x - [(1)y + x(1) \frac{dy}{dx}] + 2y \frac{dy}{dx} = 0
\]

\[
2x - y - x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0
\]

\[
\frac{dy}{dx} (2y - x) = y - 2x
\]

\[
\frac{dy}{dx} = \frac{y - 2x}{2y - x}
\]

\[
\frac{y - 2x}{2y - x} \text{ is undefined when the denominator } = 0
\]

\[
2y - x = 0
\]

\[
x = 2y
\]

Now, find \( x \) and \( y \):

\[
x = 2y
\]

\[
x^2 - xy + y^2 = 16
\]

(substitution)

\[
(2y)^2 - (2y)y + y^2 = 16
\]

\[
4y^2 - 2y^2 + y^2 = 16
\]

\[
3y^2 = 16
\]

\[
y = \pm \sqrt{\frac{16}{3}} = \text{approx. } \pm 2.31
\]

then, \( x = \pm 4.62 \) (approx.)

\((-4.62, -2.31) \text{ and } (4.62, 2.31)\)
Example: \( x^2 - \frac{4}{y^2} = x \)

Since it is a bit of effort to change equation (from implicit to explicit form), we'll use implicit differentiation.

Find \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \)

\[
\frac{dy}{dx} = -\frac{1}{8} y^3 - \frac{1}{4} xy^3.
\]

\[
\frac{d^2y}{dx^2} = \frac{1}{8} y^2 \left( y^3 + x \right) - \frac{1}{4} y^2 \left( y^3 + x \right)
\]

then, substitute \( \frac{dy}{dx} \)

\[
\frac{d^2y}{dx^2} = \frac{1}{8} y^2 \left( \frac{3y^5 (1-2x)}{64} \right) - \frac{1}{4} \left( \frac{y^3}{8} \right)
\]

\[
= \frac{3y^5 (1-2x)}{64} - \frac{y^3}{4} - \frac{3xy^5 (1-2x)}{32}
\]

\[
= \frac{3y^5 - 6xy^5 - 16y^3 - 6xy^5 + 12x^2 y^5}{64}
\]

\[
= \frac{3y^5 - 12xy^5 - 16y^3 + 12x^2 y^5}{64}
\]

Example: \( y = \frac{3}{x} \)

Find the 1st and 2nd derivatives.

Then, use implicit differentiation to verify these 1st and 2nd derivatives: \( y^5 - x^3 \)

First derivative (using the power rule)

\[
\frac{dy}{dx} = \left( \frac{3}{5} \right)x^{-2/5}
\]

Second derivative

\[
\frac{d^2y}{dx^2} = \left( -\frac{6}{25} \right)x^{-7/5}
\]

We continue to find the 2nd derivative.....

\[
\frac{dy}{dx} = \frac{3}{5} \frac{x^2}{y^4}
\]

\[
\frac{d^2y}{dx^2} = \frac{3}{5} \left( \frac{2x y^{-4} - 4y^{-5} \frac{dy}{dx} x^2}{y^2} \right)
\]

\[
\frac{d^2y}{dx^2} = \frac{3}{5} \left( \frac{2x y^{-4} - 4y^{-5} \frac{3}{5} x^{-2} y^3}{y^2} \right)
\]

\[
= \frac{3}{5} \left( \frac{2x y^{-4} - 4y^{-5} \frac{12x^4}{y^5}}{y^2} \right)
\]

\[
= \frac{3}{5} \left( \frac{2x y^{-4} - 4y^{-5} \frac{5y^4}{y^5}}{y^2} \right)
\]

\[
= \frac{3}{5} \left( \frac{2x y^{-4} - 4y^{-5} \frac{12x^4}{y^5}}{y^2} \right)
\]

First derivative

\[
\frac{dy}{dx} = \frac{3}{5} \frac{x^2}{y^4}
\]

\[
\frac{d^2y}{dx^2} = \frac{3}{5} \left( \frac{10xy^5 - 12x^4}{y^5} \right)
\]

Note: \( x^3 = y^5 \)

\[
y = x
\]

\[
= -\frac{6}{25} x \frac{5y^5 - 6x^3}{y^9}
\]

\[
= -\frac{6}{25} x \frac{5y^5 - 6y^5}{y^9}
\]

Second derivative

\[
= -\frac{6}{25} x \frac{12x^4}{y^5}
\]

\[
= -\frac{6}{25} x \frac{12x^4}{y^9}
\]
Inverse

Let's compare \( y' \) vs \( x' \)

derivative of \( y \) with respect to \( x \) vs \( \frac{dx}{dy} \) derivative of \( x \) with respect to \( y \)

Given: \( x^3 - xy + y^2 = 4 \)

Find \( y' \) or \( \frac{dy}{dx} \)

\[
3x^2 - [ (1)y + x(1) y' ] + 2y y' = 0
3x^2 - y - x y' + 2y y' = 0
-x y' + 2y y' = -3x^2 + y
y' (-x + 2y) = -3x^2 + y
y' = \frac{-3x^2 + y}{(-x + 2y)} \quad \text{Reciprocals}
\]

Find \( x' \) or \( \frac{dx}{dy} \)

\[
3x^2 x' - [ (1)x' y + x(1) ] + 2y = 0
3x^2 x' - x' y - x + 2y = 0
3x^2 x' - x' y = x - 2y
x' (3x^2 - y) = x - 2y
x' = \frac{x - 2y}{(3x^2 - y)} = \frac{-x + 2y}{-3x^2 + y}
\]

To find \( dx/dy \), we insert \( x \) whenever taking the derivative of \( x' \)
Implicit Differentiation: Word Problem Examples

1) A 25-foot ladder is leaning against a wall. If the top of the ladder is slipping down the wall at a rate of 2 feet/second, how fast will the bottom be moving away from the wall when the top is 20 feet above the ground?

Step 1: Draw diagram, list variables and formulas

\[ x^2 + y^2 = 625 \text{ ft}^2 \] (Pythagorean theorem)

\[ \frac{dy}{dt} = -2 \text{ ft/sec} \] (change of y with respect to time)

moving away from the wall
\[ \frac{dx}{dt} = ? \] (change of x with respect to time)

Step 2: Set up equation and use implicit differentiation.

\[ 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \] derivative with respect to time

Substitute and solve:

\[ 2x \frac{dx}{dt} + (20 \text{ ft})(-2 \text{ ft/sec}) = 0 \]

\[ \frac{x^2 + y^2}{(x)^2 + (20 \text{ ft})^2} = 625 \text{ ft}^2 \]

When \( y = 20 \text{ ft} \), \( x = 15 \) feet

\[ 2(15 \text{ ft}) \frac{dx}{dt} + (-80 \text{ ft}^2/\text{sec}) = 0 \]

\[ 30 \text{ ft} \frac{dx}{dt} = 80 \text{ ft}^2/\text{sec} \]

\[ \frac{dx}{dt} = \frac{80 \text{ ft}^2/\text{sec}}{30 \text{ ft}} = 2.67 \text{ ft/sec} \]

Step 3: Check answer

From 22 to 20 feet (one second), the ladder moved out 3.13 feet

From 21 to 19 feet (one second), the ladder moved out 2.69 feet...

From 20 to 18 feet (one second), the ladder moved 2.35 feet...

2.67 feet per second is a reasonable answer! 

Important note: We're seeking \( \frac{dx}{dt} \), (the change of x with respect to time).

Simply taking the derivative of \( y = \sqrt{625 - x^2} \)

\[ \frac{1}{2} (625 - x^2)^{-1/2} (-2x) = \frac{-x}{\sqrt{625 - x^2}} \]

shows us \( \frac{dy}{dx} \), (the change of y with respect to x)
2) Oil erupts from a ruptured tanker, spreading in a circle whose area increases at a constant rate of 6 square miles per hour. How fast is the radius of the spill increasing when the area is $9\pi^2$ square miles?

Step 1: Draw a diagram, list variables, and consider formulas

- Spill area: $A = \pi r^2$
- "area is increasing at a rate of 6 square miles per hour" $\frac{dA}{dt} = 6 \text{ miles}^2 \text{ hour}^{-1}$
- "how fast is the radius of the spill increasing?" $\frac{dr}{dt} = ?$

When area is $9\pi^2$ sq. miles, the radius is 3 miles.

Step 2: Implicit differentiation

Take derivative with respect to $t$

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

Plug in values and solve

$6 \text{ miles}^2 \text{ hour}^{-1} = 2\pi (3 \text{ miles}) \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{6 \text{ miles}^2 \text{ hour}^{-1}}{2\pi (3 \text{ miles})} = \frac{1}{\pi} \text{ miles/hour}$

(or, $0.318 \text{ miles/hour}$)

When area of spill is $9\pi^2$ square miles, the radius is increasing at $0.318$ miles per hour.

Step 3: Verify Answer

- Area: $A = (3.14)r^2$
- 12 sq miles: $12 = (3.14)r^2$, $r = 1.95 \text{ miles}$ (one hour later)
- 18 sq miles: $18 = (3.14)r^2$, $r = 2.39 \text{ miles}$ (one hour later)
- 24 sq miles: $24 = (3.14)r^2$, $r = 2.76 \text{ miles}$
- Radius changed 0.33 miles in one hour
- 30 sq miles: $30 = (3.14)r^2$, $r = 3.09 \text{ miles}$ (one hour later)

When area changes, radius changes.
Find $\frac{dy}{dx}$ if $y = x^x$

Answer: $\ln y = x \ln(x)$  logarithm power rule

$$\frac{dy}{dx}: \quad \frac{1}{y} \frac{dy}{dx} = (1) \ln x + x \frac{1}{x}$$

$$\frac{1}{y} \frac{dy}{dx} = \ln x + 1$$

$$\frac{dy}{dx} = y(\ln x + 1)$$

$$= x^x (\ln x + 1)$$
**Implicit Differentiation & Inverse Trig Derivatives**

**Example:** \( y = \sin^{-1} x \)  
What is \( \frac{dy}{dx} \)?

**Step 1:** Change the inverse trig term

\[
\sin y = \sin (\sin^{-1} x)
\]

\[
\sin(y) = x
\]

**Step 2:** "Draw the triangle"

Sine \( y = \frac{\text{opposite}}{\text{hypotenuse}} \)

Pythagorean Theorem

\[
a^2 + b^2 = c^2
\]

**Step 3:** Use implicit differentiation to find \( \frac{dy}{dx} \)

\[
\cos(y) \frac{dy}{dx} = 1
\]

\[
\frac{dy}{dx} = \frac{1}{\cos(y)}
\]

using the triangle,

\[
\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}
\]

**Example:** \( y = \tan^{-1} (6x) \)  
Find the derivative.

**Step 1:** Change the inverse trig term

\[
\tan y = \tan(\tan^{-1} (6x))
\]

\[
\tan(y) = 6x
\]

**Step 2:** "Draw the triangle"

Tangent \( y = \frac{\text{opposite}}{\text{adjacent}} \)

Pythagorean Theorem

\[
a^2 + b^2 = c^2
\]

**Step 3:** Use implicit differentiation to find \( \frac{dy}{dx} \)

\[
\sec^2 (y) \frac{dy}{dx} = 6
\]

\[
\frac{dy}{dx} = \frac{6}{\sec^2(y)}
\]

using the triangle,

\[
\frac{dy}{dx} = \frac{6}{\sqrt{1 + 36x^2}} = \frac{6}{1 + 36x^2}
\]
Practice Quiz→

"In this one, you'll revisit the power, product, and chain rules."

"Oh, and here's a hint: remember to differentiate with respect to x."

POP QUIZ!! Implicit Differentiation

\[ \sin(xy) + x^2y^3 = z \]

Find \( \frac{dy}{dx} \)

"Respect x? I don't even like x. Why do I have to respect it?"

"Do you know what the word 'implicit' implies?"

"This implicit one is a *&%$#-ing nightmare!"

... A class full of implicit and explicit expressions!

Explicit language

Implicit Differentiation

Calculators
1) If $y = \sin(x)\cos(y)$, then at $(\pi, 0)$, $\frac{dy}{dx} = \ldots$

   a) $-1$
   b) $0$
   c) $1$
   d) $\pi$
   e) $2\pi$

2) If $x^2 + 2y^2 = 22$, what is the behavior of the graph at $(-2, 3)$

   a) increasing, concave up
   b) increasing, concave down
   c) decreasing, concave up
   d) decreasing, concave down
   e) increasing, point of inflection

3) A plane flies 6 miles high above the ground. At the moment, it is directly 10 miles from an airport tower, and it is approaching the tower at a rate of 400 miles per hour.

   How fast is the plane traveling?
SOLUTIONS

1) If \( y - \sin(x)\cos(y) \), then \( (\pi, 0) \) \( \frac{dy}{dx} \) is:

- a) -1
- b) 0
- c) 1
- d) \( \pi \)
- e) 2 \( \pi \)

\[
\begin{align*}
\frac{dy}{dx} &= \cos(x)\cos(y) - \sin(y) \frac{dy}{dx} \sin(x) \\
to \ find \ IROC \ at \ point, \ substitute \ (\pi, 0) \\
\frac{dy}{dx} &= \cos(\pi)\cos(0) - \sin(0) \frac{dy}{dx} \sin(\pi) \\
&= (-1)(1) + (0) \frac{dy}{dx} (0) = -1
\end{align*}
\]

2) If \( x^2 + 2y^2 = 22 \), what is the behavior of the graph at \((-2, 3)\):

- a) increasing, concave up
- b) increasing, concave down
- c) decreasing, concave up
- d) decreasing, concave down
- e) increasing, point of inflection

Notice, the graph is an ellipse!

3) A plane flies 6 miles high above the ground.
At the moment, it is directly 10 miles from an airport tower, and it is approaching the tower at a rate of 400 miles per hour.

How fast is the plane traveling?

\[
\begin{align*}
x &= \text{horizontal distance} \\
d &= \text{direct distance} \\
h &= \text{height above ground}
\end{align*}
\]

\[
\begin{align*}
x^2 + h^2 &= d^2 \\
2x \frac{dx}{dt} + 2h \frac{dh}{dt} &= 2d \frac{dd}{dt} \\
16 \frac{dx}{dt} + 12(0) &= 20(-400)
\end{align*}
\]

\[
\frac{dx}{dt} = 500 \text{ m/h}
\]

\[
\frac{dd}{dt} = -400 \text{ m/h}
\]

\[
\frac{dh}{dt} = 0
\]
Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers

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