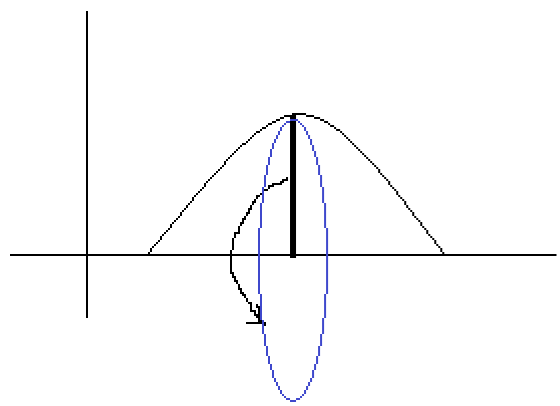
Calculus: Integrals, Area, and Volume

Notes, Examples, Formulas, and Practice Test (with solutions)

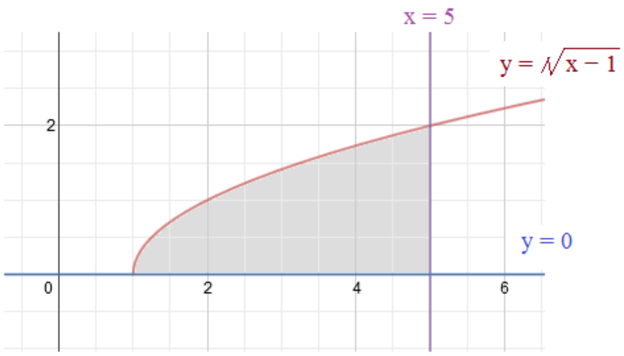


Topics include definite integrals, area, “disc method”, volume of a solid from rotation, and more.

Mathplane.com

We've learned that the area under a curve can be found by evaluating a definite integral.

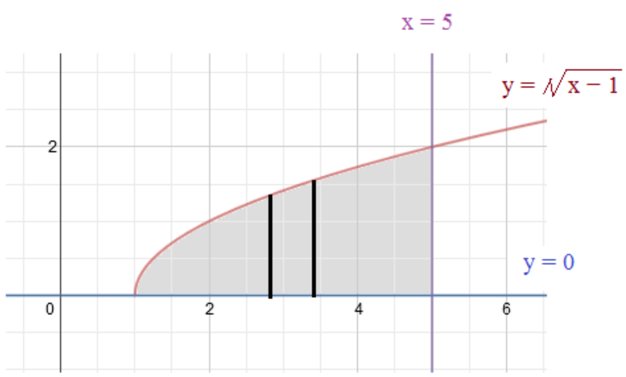
Example: Find the area in the region bounded by $x = 5$
$y = 0$
$y = \sqrt{x - 1}$



Area *under* the curve:
(Shaded Area)

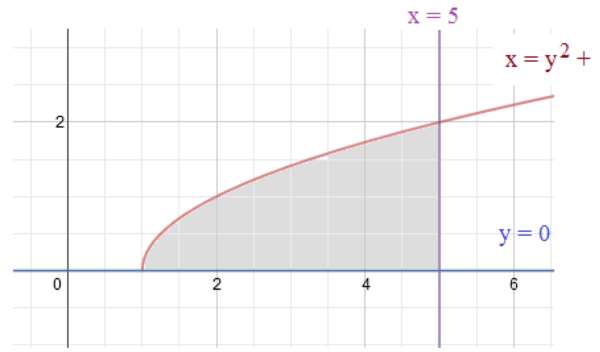
$$\int_0^5 \sqrt{x-1}\, dx - \int_0^5 0\, dx$$

$$\frac{2}{3}(x-1)^{\frac{3}{2}}\Big|_0^5 - 0 = \frac{16}{3}$$



The area was found by taking vertical partitions.

$$\text{Area} = \int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

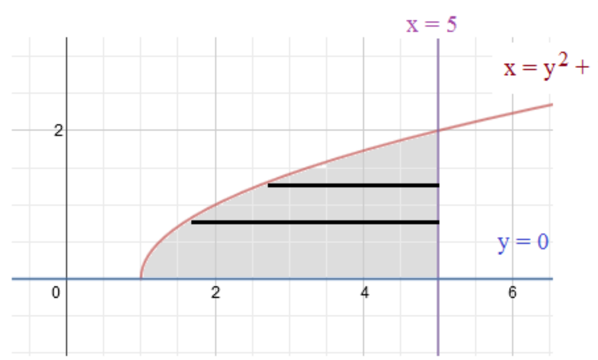


Area *right* of the curve:
(Shaded Area)

$$\int_0^2 5\, dy - \int_0^2 y^2 + 1\, dy$$

$$5y\Big|_0^2 - \left(\frac{y^3}{3} + y\Big|_0^2\right)$$

$$10 - 0 - \left(\frac{8}{3} + 2 - 0 - 0\right) = \frac{16}{3}$$



The area was found by taking horizontal partitions.

$$\text{Area} = \int_c^d f(y)\, dy = \lim_{n \to \infty} \sum_{i=1}^{n} f(y_i)\,\Delta y$$

Area = (length)(width)

Volume = (length)(width)(depth) = Area(depth)

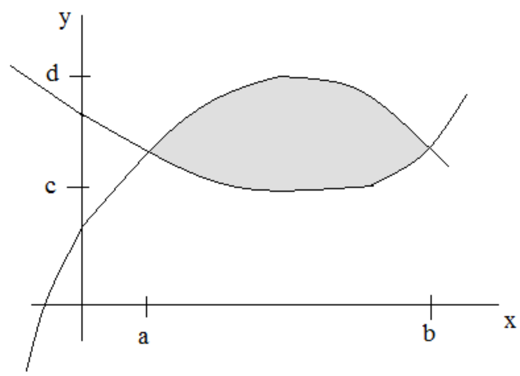
Since

$$\text{Area} = \int_a^b f(x)\ dx \quad \text{or} \quad \int_c^d f(y)\ dy$$

then,

$$\text{Volume} = \int_a^b A(x)\ dx \qquad \text{Volume} = \int_c^d A(y)\ dy$$

where A(x) or A(y) is a cross-section area of a solid



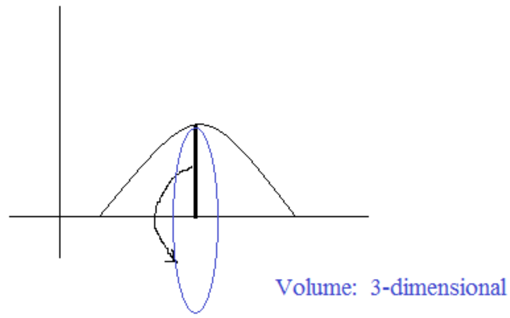
"Disc Method"

If we rotate each segment, we get a sequence of circles...

(Circle) Area = π (radius)^2

So, the volume of the solid will be the sum of all the circles!

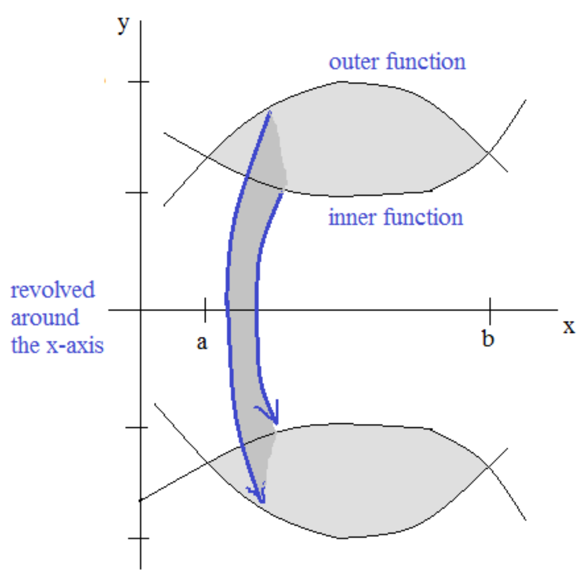
(The *radius* (length) *of each circle* is determined by the output of the function)



$$\text{Volume} = \int_a^b \pi\ (\text{function})^2\ dx$$

And, if you get a "ring",

$$\text{Volume} = \int_a^b \pi\ (\textit{outer}\ \text{function})^2\ dx - \int_a^b \pi\ (\textit{inner}\ \text{function})^2\ dx$$



Volume of a cone from rotated line segment

Example: If a portion of the line $y = \frac{1}{2}x$ lying in Quadrant I is rotated around the x-axis, a solid cone is generated.
Find the volume of the cone extending from x = 0 to x = 6.

The length (height) of the cone will extend from 0 to 6

$$\int_0^6 \quad dx$$

The area from the segments will be from the function $\frac{1}{2}x$

(These are the 'radii')

$$\int_0^6 \frac{1}{2}x \; dx$$

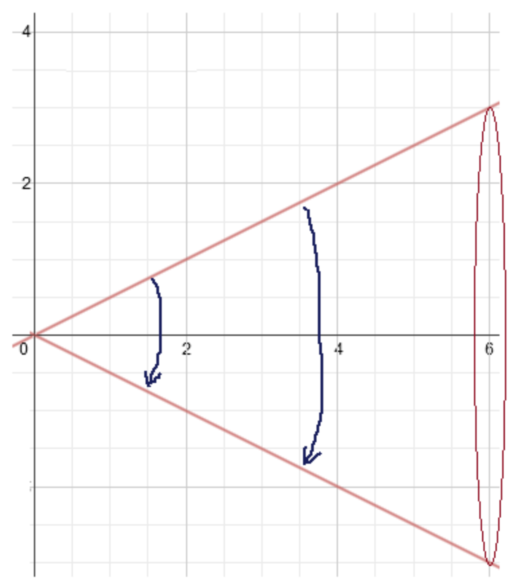
And, the volume of the solid from rotation (revolution) will be from the total area of the segments (radii)

(These are the round 'discs')

$$\int_0^6 \pi\left(\frac{1}{2}x\right)^2 dx$$

$$\pi \int_0^6 \frac{x^2}{4}\, dx$$

$$\pi \left.\frac{x^3}{12}\right|_0^6 = 18\pi$$



Quick Check: Volume of a cone: $\frac{1}{3}\pi\,(\text{radius})^2\,(\text{height})$

This (sideways) cone: Radius = 3
Height = 6

Volume = $\frac{1}{3}\pi\,(3)^2\,(6) = 18\pi$

Calculus and Area Rotation

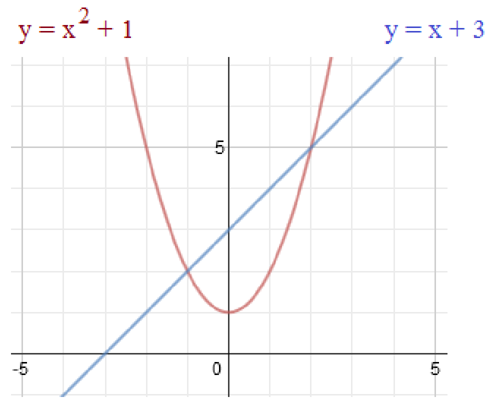
Find the volume of the figure where the cross-section area is bounded by

$y = x^2 + 1$

$y = x + 3$

and revolved around the x-axis.

Step 1: Draw a sketch



Step 2: Determine the span of the integral

$y = x^2 + 1$

$y = x + 3$

$x^2 + 1 = x + 3$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0 \qquad x = -1, 2$

The boundaries of the area are [-1, 2]

Step 3: Write the integral(s)

The bounded area will revolve around the x-axis

$$\int_{-1}^{2} \pi (x+3)^2 \, dx - \int_{-1}^{2} \pi (x^2+1)^2 \, dx$$

Area under the line from -1 to 2

Area under the curve from -1 to 2

Step 4: Evaluate the integrals

$$\int_{-1}^{2} \pi (x+3)^2 \, dx - \int_{-1}^{2} \pi (x^2+1)^2 \, dx$$

$$\pi \int_{-1}^{2} x^2 + 6x + 9 \, dx - \pi \int_{-1}^{2} x^4 + 2x^2 + 1 \, dx$$

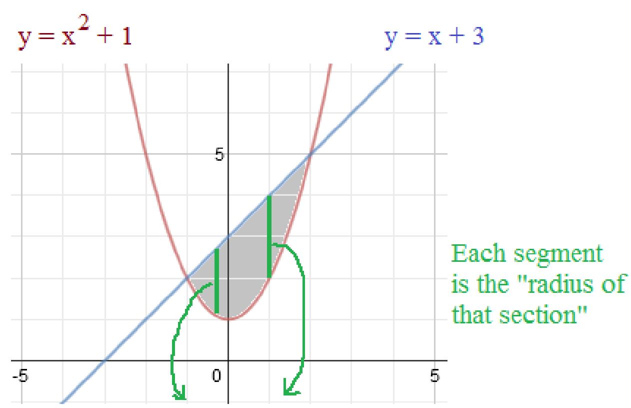
$$\pi \left(\frac{x^3}{3} + 3x^2 + 9x \Big|_{-1}^{2} \right) - \pi \left(\frac{x^5}{5} + \frac{2x^3}{3} + x \Big|_{-1}^{2} \right)$$

$$\pi \left(32\tfrac{2}{3} - (-6\tfrac{1}{3}) \right) - \pi \left(\frac{206}{15} - \left(\frac{-28}{15} \right) \right)$$

$$39\pi - \frac{234}{15}\pi =$$

$$39\pi - \frac{78}{5}\pi$$

$$\boxed{\frac{117}{5}\pi}$$



NOTE: Volume = $\int_a^b A(x)\, dx$

$\int \pi r^2$ "where r is the function that is being revolved"

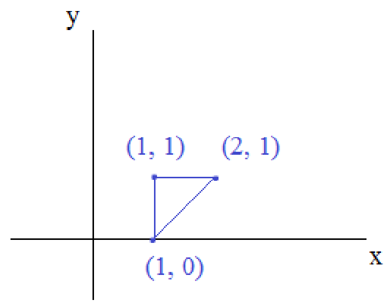
Calculus and Volume (of solids from rotation)

A triangle with vertices (1, 0) (2, 1) and (1, 1)

is rotated around the *y-axis*.

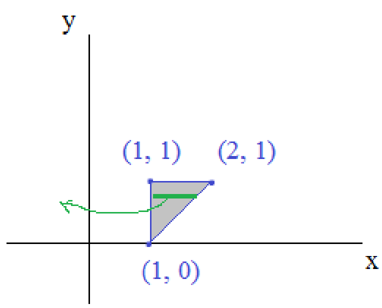
What is the volume of the solid?

Step 1: Draw a sketch



Step 2: Determine the boundaries of the integral

Since the rotation is *around the y-axis*,
the boundaries will be between y = 0 and y = 1



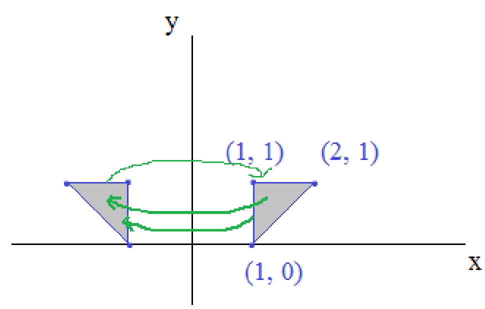
Step 3: Write the integrals

The line connecting (1, 0) and (2, 1)
is $y = x - 1$

or, $x = y + 1$

And, the line connecting (1, 0) and (1, 1) is $x = 1$

$$\int_0^1 \pi (y+1)^2 \, dy \; - \; \int_0^1 \pi (1)^2 \, dy$$



Step 4: Evaluate integrals to find volume

$$\int_0^1 \pi (y+1)^2 \, dy \; - \; \int_0^1 \pi (1)^2 \, dy$$

$$\int_0^1 \pi (y^2 + 2y + 1) \, dy \; - \; \int_0^1 \pi \, dy$$

$$\pi \left(\frac{y^3}{3} + y^2 + y \,\Bigg|_0^1 \right) - \pi \left(y \,\Bigg|_0^1 \right)$$

$$2\tfrac{1}{3}\pi - \pi = \boxed{\frac{4}{3}\pi}$$

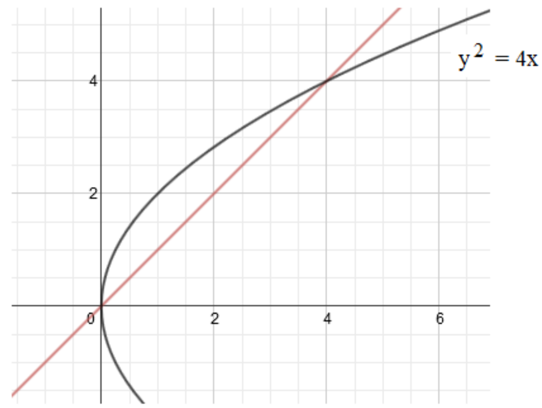
NOTE: Volume = $\int_a^b A(y) \, dy$ ("integral of Area")

$\int \pi r^2$ "where r is the function that is being rotated"

Example: Find the volume of the solid formed by the region bounded by

$$y = x$$
$$y^2 = 4x$$

a) rotated around the x-axis
b) rotated around the y-axis
c) rotated around $x = 4$



a) Since the region is rotated around the x-axis, we'll use 'vertical partitions'.
The left boundary will be $x = 0$
and the right boundary will be $x = 4$

The upper boundary will be $y^2 = 4x$ → $y = 2\sqrt{x}$

The 2-dimensional area of the region would be the integral

$$\int_0^4 2\sqrt{x}\,dx \; - \; \int_0^4 x\,dx \; =$$

area from curve to x-axis — *area from line to x-axis* — *area of enclosed region*

But, the volume adds another dimension... Each segment in the area is rotated to form a disc (circle)
(and, the segments are the radii of all the discs in the solid!)

$$\int_0^4 \pi(2\sqrt{x})^2\,dx \; - \; \int_0^4 \pi(x)^2\,dx$$

Area of circle = $\pi\,(\text{radius})^2$

$$\pi\int_0^4 4x - x^2\,dx$$

$$\pi\left(2x^2 - \frac{x^3}{3}\right)\Bigg|_0^4 = \left(32 - \frac{64}{3}\right)\pi = 10\tfrac{2}{3}$$

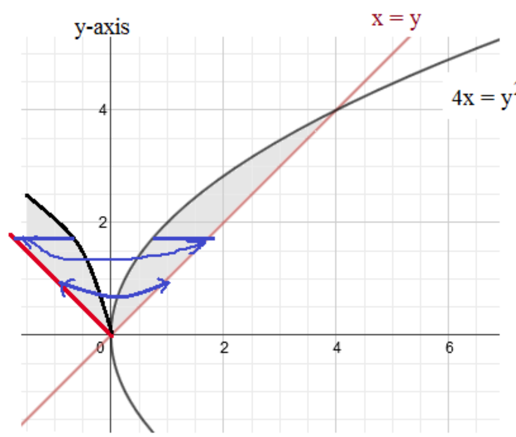
$$\text{Volume} = \int_a^b \pi\,(\text{function})^2\,dx$$ ('sum of vertical discs')

b) Since this is rotated around the y-axis, we'll use 'horizontal partitions'
The lower boundary will be $y = 0$.
And, the upper boundary will be $y = 4$.

The right boundary ('outer radius') will be $x = y$
And, the left boundary ('inner radius') will be $x = \frac{y^2}{4}$ ← $y^2 = 4x$

$$\text{Volume} = \int_c^d A(y)\,dy$$

where $A(y)$ is a cross-section area of a solid



$$\int_0^4 \pi(y)^2\,dy \; - \; \int_0^4 \pi\left(\frac{y^2}{4}\right)^2 dy$$

volume of line revolved around y-axis (forms a cone) — *volume of curve revolved around y-axis (forms a 'funnel')*

$$\pi\int_0^4 y^2\,dy \;-\; \pi\int_0^4 \frac{y^4}{16}\,dy$$

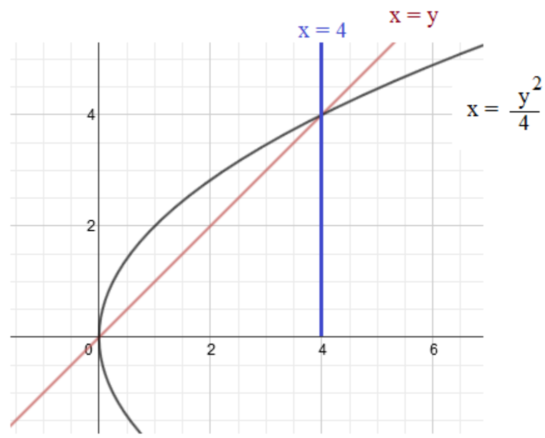
volume of outer part (cone) *volume of inner part (funnel)*

$$\pi\,\frac{y^3}{3}\Big|_0^4 - \pi\,\frac{y^5}{80}\Big|_0^4 = \frac{64}{3}\pi - \frac{64}{5}\pi = \boxed{\frac{128}{15}\pi}$$

c) In this case, the region is rotated around x = 4 (instead of an axis)
We'll use 'horizontal partitions' (*dy*) from y = 0 to y = 4

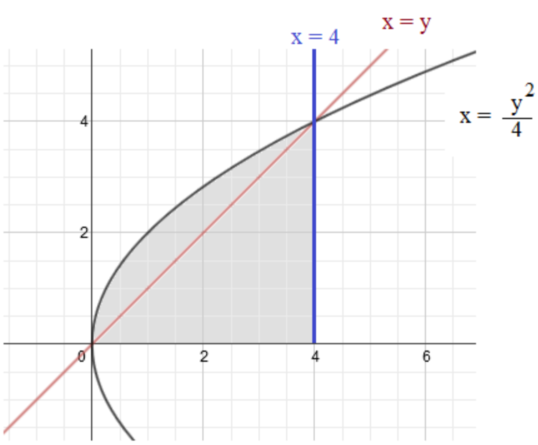
The volume integrals are:

$$V = \int_0^4 \pi\left(4 - \frac{y^2}{4}\right)^2 dy \;-\; \int_0^4 \pi\,(4 - y)^2\,dy$$

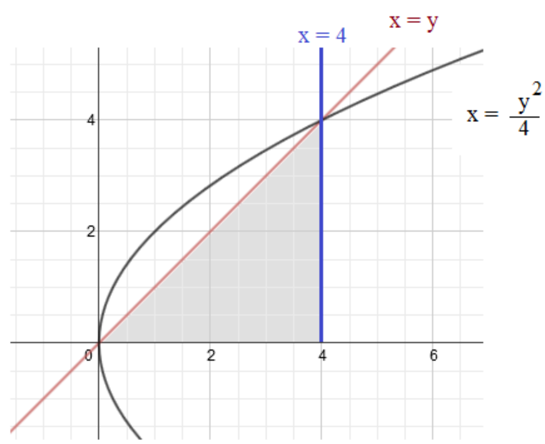


Observe where the area functions came from:
(the difference is the bounded region!)

The shaded area is $\int_0^4 4 - \frac{y^2}{4}\,dy$



The shaded area is $\int_0^4 4 - y\;dy$



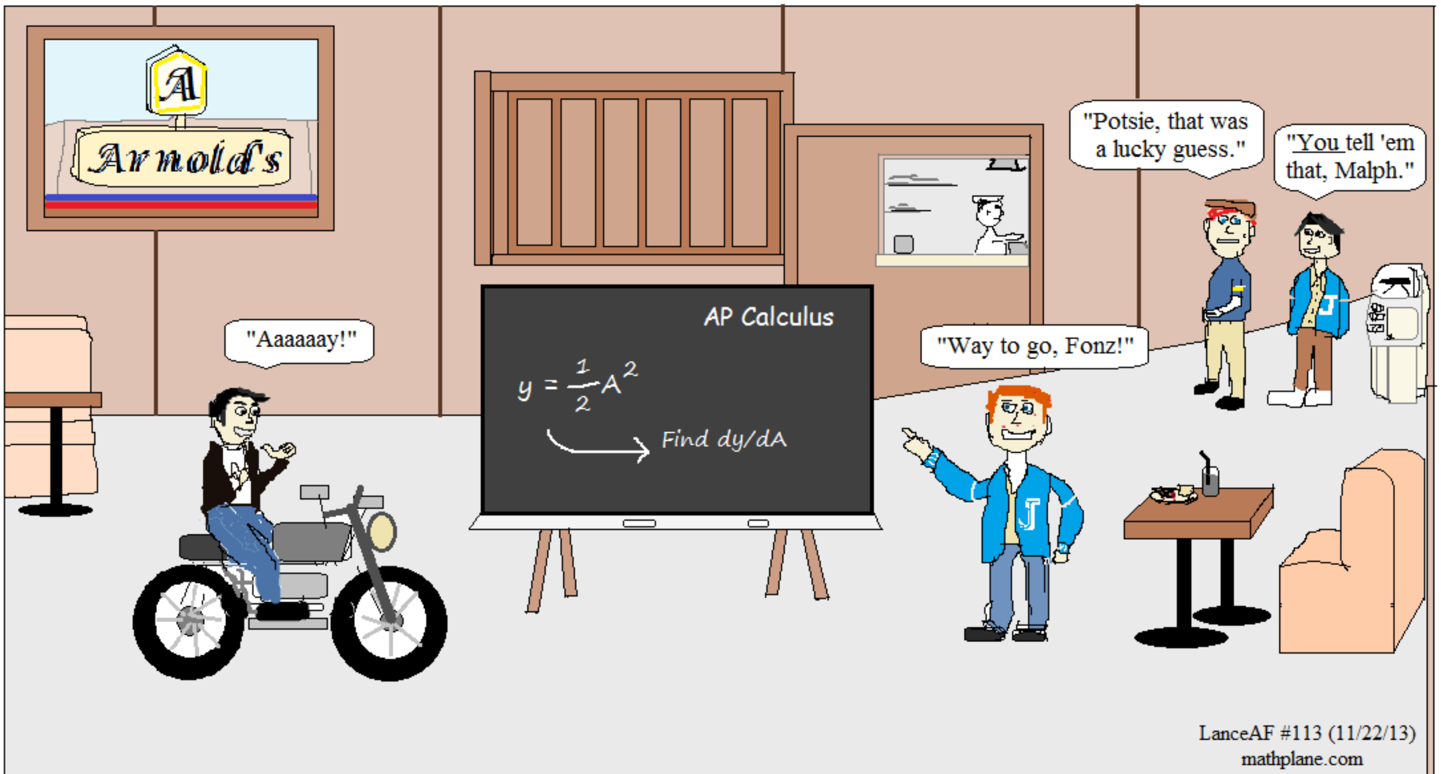
$$\text{Volume} = \int_0^4 \pi\left(4 - \frac{y^2}{4}\right)^2 dy \;-\; \int_0^4 \pi\,(4 - y)^2\,dy$$

$$\pi\int_0^4 16 - 2y^2 + \frac{y^4}{16}\;dy \;-\; \pi\int_0^4 16 - 8y + y^2\;dy$$

$$\text{Volume} = \int_c^d \pi\,(\text{function})^2\,dy$$

(sum of the horizontal discs)

$$\pi\int_0^4 \frac{y^4}{16} - 3y^2 + 8y\;dy \;\longrightarrow\; \pi\left(\frac{y^5}{80} - y^3 + 4y^2\Big|_0^4\right) = (12.8 + 64 + 64)\pi = \boxed{12.8\pi}$$



Happy Days

Despite Richie's help, Fonzie dropped out of Calculus.
(... although he did have some success with velocity and acceleration!)

Practice Test -→

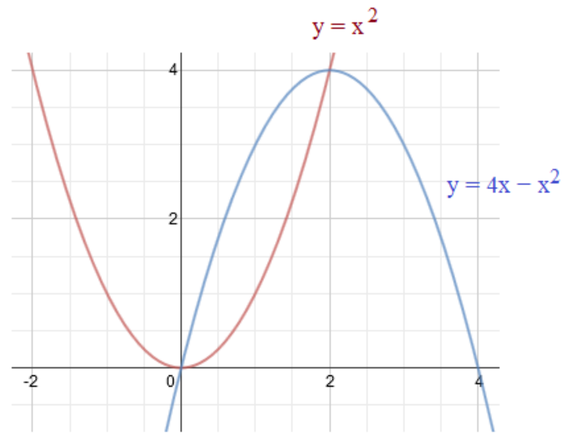
Volume of Solids Practice Test

1) Find the volume of the solid formed by revolving the region bounded by

$$y = x^2 \quad \text{and} \quad y = 4x - x^2$$

a) about the x-axis

b) about the line $y = 6$



2) Given the area bounded by $y = \sqrt{x}$

$y = 2$

$x = 0$

Find the volume of the solid from rotation

a) about the x-axis

b) about the y-axis

c) around $y = 2$

Volume of Solids Practice Test

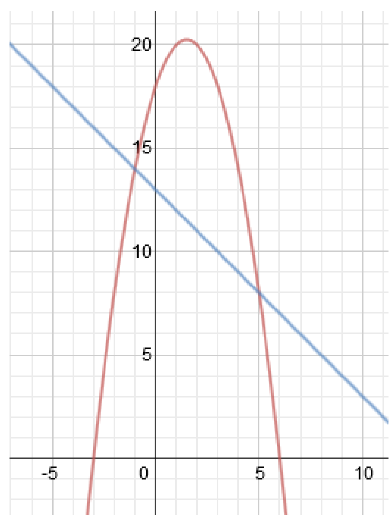
3) The left half of the ellipse $9x^2 + 25y^2 = 225$ is revolved around the y-axis to form a spheroid. Find its volume.

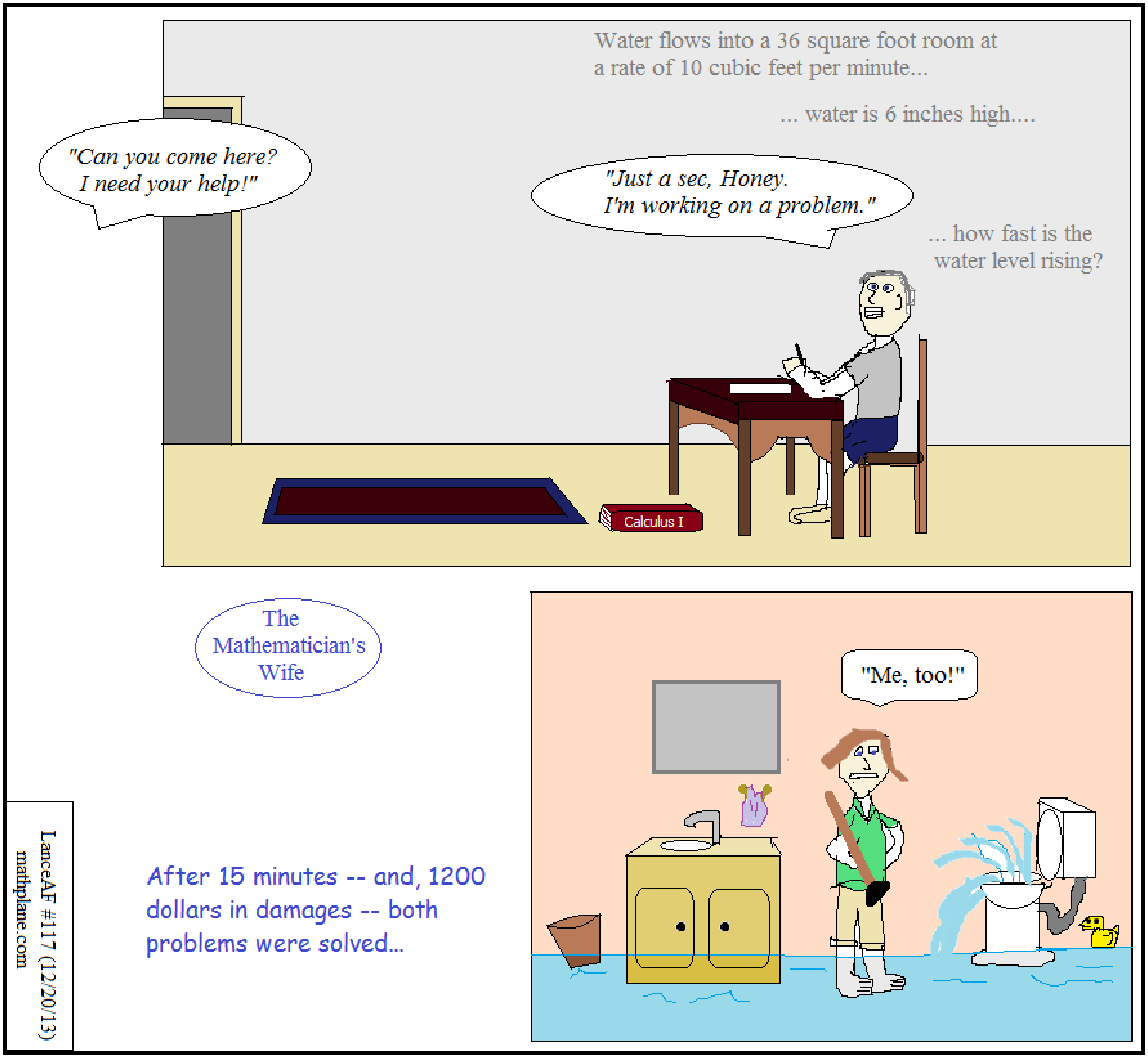
4) Find the volume of the solid formed by the region bounded by

$$y = -x^2 + 3x + 18$$

$$x + y = 13$$

and revolved around the x-axis




Water flows into a 36 square foot room at a rate of 10 cubic feet per minute...
... water is 6 inches high....
"Can you come here? I need your help!"
"Just a sec, Honey. I'm working on a problem."
... how fast is the water level rising?
Calculus I
The Mathematician's Wife
"Me, too!"
After 15 minutes -- and, 1200 dollars in damages -- both problems were solved...
LanceAF #117 (12/20/13)
mathplane.com

ANSWERS -→

Volume of Solids Practice Test

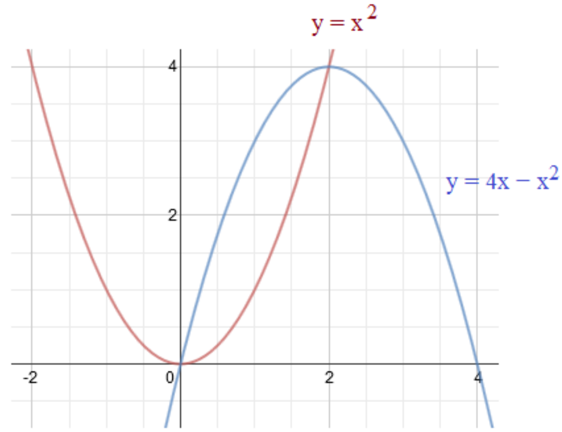
SOLUTIONS

1) Find the volume of the solid formed by revolving the region bounded by

$$y = x^2 \quad \text{and} \quad y = 4x - x^2$$

a) about the x-axis

b) about the line $y = 6$



a) The graph shows the 2 parabolas intersecting at $x = 0$ and $x = 2$

(algebraically) $x^2 = 4x - x^2$

$2x^2 - 4x = 0$ $\qquad x = 0, 2$

$2x(x - 2) = 0$

Since the rotation is about the x-axis,

the *outer* (radius) boundary will be $4x - x^2$

the *inner* (radius) boundary of the solid will be x^2

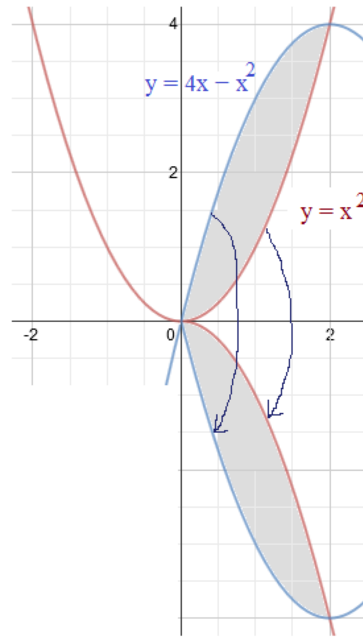
$$\int \pi (4x - x^2)^2\, dx - \int \pi (x^2)^2\, dx$$

And, the intersections indicate the ends of definite integral

$$\int_0^2 \pi (4x - x^2)^2\, dx - \int_0^2 \pi (x^2)^2\, dx$$

$$\pi \int_0^2 16x^2 - 8x^3 + \not{x^4}\, dx - \pi \int_0^2 \not{x^4}\, dx$$

$$\pi \left(\frac{16x^3}{3} - 2x^4 \Big|_0^2 \right) = \boxed{10\tfrac{2}{3}\pi}$$



b) Since the rotation is about the line $y = 6$,

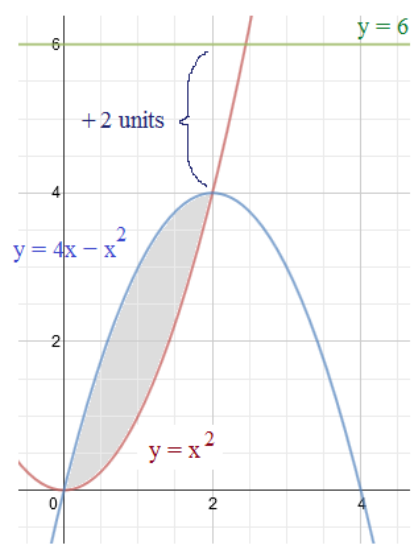
the *outer* (radius) boundary will be $x^2 + 2$

the *inner* (radius) boundary of the solid will be $(4x - x^2) + 2$

$$\int_0^2 \pi (x^2 + 2)^2\, dx - \int_0^2 \pi (-x^2 + 4x + 2)^2\, dx$$

$$\pi \int_0^2 \not{x^4} + 4x^2 + \not{4}\, dx - \pi \int_0^2 \not{x^4} - 8x^3 + 12x^2 + 16x + \not{4}\, dx$$

$$\pi \int_0^2 8x^3 - 8x^2 - 16x\, dx = \pi\ 2x^4 - \frac{8}{3}x^3 - 8x^2 \Big|_0^2 = \boxed{\frac{64}{3}\pi}$$



Volume of Solids Practice Test

SOLUTIONS

2) Given the area bounded by $y = \sqrt{x}$

$y = 2$

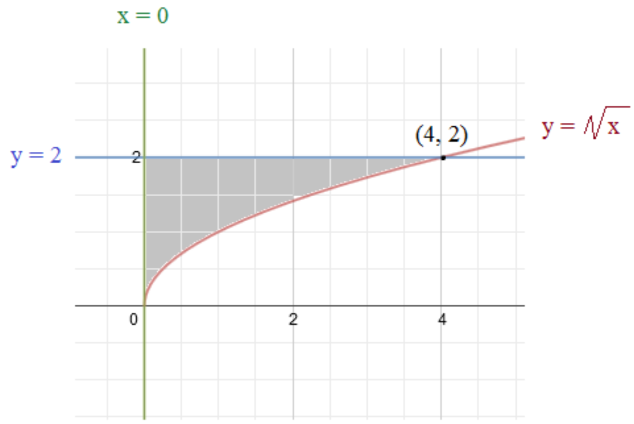
$x = 0$

Find the volume of the solid from rotation

a) about the x-axis

b) about the y-axis

c) around $y = 2$



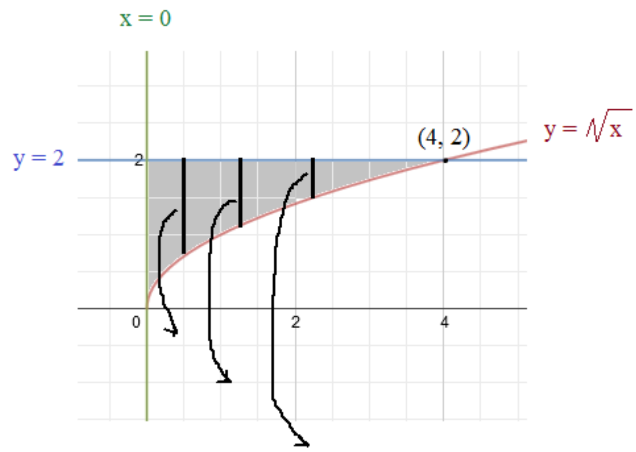
a) Since the rotation (revolution) is about the x-axis, the outer radius will be $y = 2$, and the inner radius will be $y = \sqrt{x}$

Then, the endpoints (or limits of integration) will be 0 and 4

$$\int_0^4 \pi (2)^2 \, dx - \int_0^4 \pi (\sqrt{x})^2 \, dx$$

$$\pi \, [4x] \Big|_0^4 - \pi \left[\frac{x^2}{2} \right] \Big|_0^4$$

$$16\pi - 8\pi = \boxed{8\pi}$$



b) Since the revolution (rotation) is about the y-axis, we need to rewrite the expression:

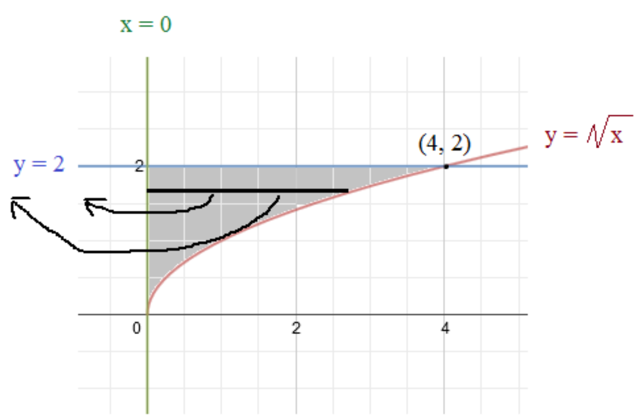
$y = \sqrt{x} \longrightarrow x = y^2$

Then, the endpoints (limits of integration) will be 0 and 2

$$\int_0^2 \pi (y^2)^2 \, dy$$

$$\pi \int_0^2 y^4 \, dy$$

$$\pi \frac{y^5}{5} \Big|_0^2 = \boxed{\frac{32}{5}\pi}$$



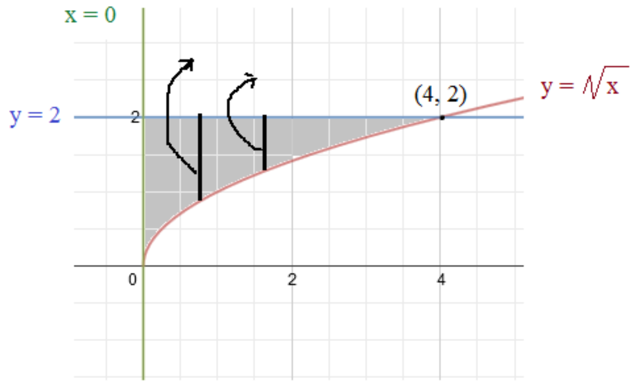
c) Since the rotation is around $y = 2$, the radius will come from the area between $y = 2$ and $y = \sqrt{x}$

$$\int_0^4 \pi \left(2 - \sqrt{x}\right)^2 \, dx$$

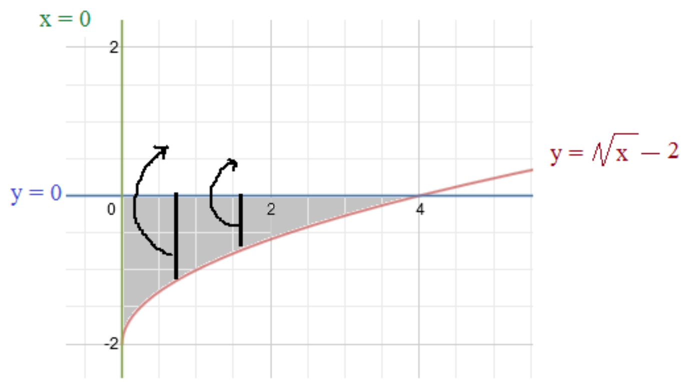
$$\pi \int_0^4 4 - 4\sqrt{x} + x \, dx$$

$$\pi \left[4x - \frac{4x^{3/2}}{3/2} + \frac{x^2}{2} \right] \Bigg|_0^4$$

$$16\pi - \frac{64}{3}\pi + 8\pi = \frac{8}{3}\pi$$



**The volume of solid would be the same if we *shift the area down 2* units and *rotate it around the x-axis*!



$$\int_0^4 \pi \left(\sqrt{x} - 2\right)^2 \, dx$$

$$\pi \int_0^4 x - 4\sqrt{x} + 4 \, dx$$

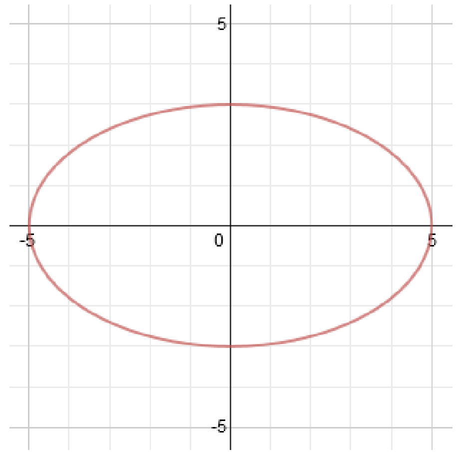
$$\pi \left(\frac{x^2}{2} - \frac{4x^{3/2}}{3/2} + 4x \Bigg|_0^4 \right)$$

$$\pi \left(8 - \frac{64}{3} + 16 - 0 - 0 - 0 \right) = \frac{8}{3}\pi$$

3) The left half of the ellipse $9x^2 + 25y^2 = 225$ is revolved around the y-axis to form a spheroid. Find its volume.

SOLUTIONS

First, let's graph the ellipse:



$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

We can see the left half extends up/down from 3 to -3 and the left half from -5 to 0...

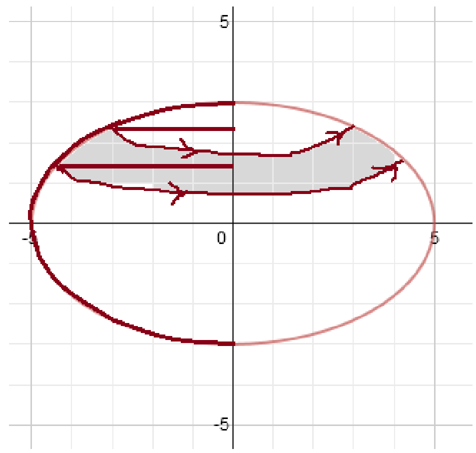
Since the half ellipse is revolved around the y-axis,

$$\int \qquad dy$$

Since the left half is symmetric over the x-axis, we'll find the volume for the top half (and then double it)

$$\int_0^3 \qquad dy$$

"Disc method"



$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

$$x^2 = 25 - \frac{25y^2}{9} \qquad x = \sqrt{25 - \frac{25y^2}{9}}$$

$$\int_0^3 \pi \left(25 - \frac{25y^2}{9}\right) dy$$

$$\text{Volume} = \int \pi\, f(y)^2\, dy$$

$$\pi \left(25y - \frac{25y^3}{27} \Bigg|_0^3 \right) = 50\pi$$

Therefore, the full spheroid (top half and bottom half) is $\boxed{100\pi}$

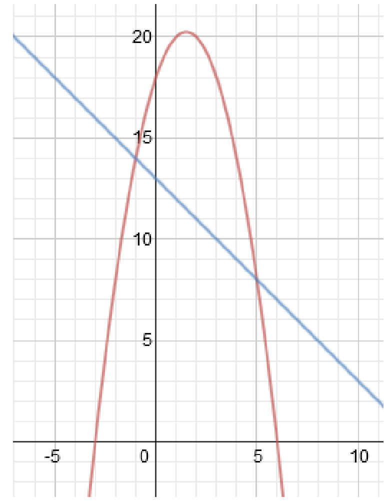
4) Find the volume of the solid formed by the region bounded by

$$y = -x^2 + 3x + 18$$

$$x + y = 13$$

and revolved around the x-axis

SOLUTIONS



When we graph the equations, we observe an upside down parabola that is intersected by a line.

The boundary (from left to right) are the points of intersection.

$$\begin{cases} y = -x^2 + 3x + 18 \\ y = -x + 13 \end{cases} \quad \text{substitution}$$

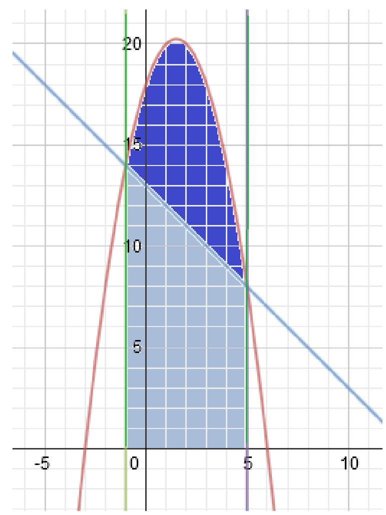
$$-x + 13 = -x^2 + 3x + 18 \quad \text{collect terms}$$

$$x^2 - 4x - 5 = 0 \quad \text{factor}$$

$$(x - 5)(x + 1) = 0 \quad \text{solve}$$

$x = -1$ and $x = 5$

The outer radius will be from the parabola
The inner radius will be from the line



$$\int_{-1}^{5} -x^2 + 3x + 18 \; dx \; - \; \int_{-1}^{5} -x + 13 \; dx$$

area below the parabola — area below the line

$$\int_{-1}^{5} \pi \left(-x^2 + 3x + 18\right)^2 dx \; - \; \int_{-1}^{5} \pi \left(-x + 13\right)^2 dx$$

discs from the parabola — discs from the line

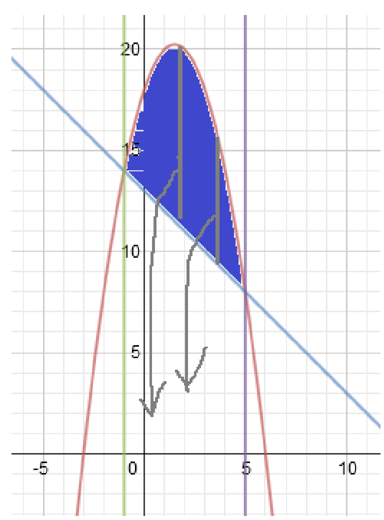
$$\pi \int_{-1}^{5} x^4 - 6x^3 - 27x^2 + 108x + 324 \; dx \; - \; \pi \int_{-1}^{5} x^2 - 26x + 169 \; dx$$

$$\pi \int_{-1}^{5} x^4 - 6x^3 - 28x^2 + 134x + 155 \; dx$$

$$\pi \left(\frac{x^5}{5} - \frac{3x^4}{2} - \frac{28x^3}{3} + 67x^2 + 155x \Bigg|_{-1}^{5} \right)$$

$$\pi \left(625 - 937.5 - 1166\tfrac{2}{3} + 1675 + 775 - \left(\frac{-1}{5} - \frac{3}{2} - \frac{-28}{3} + 67 - 155 \right) \right)$$

$$970.83 - (-80.36) = \boxed{1051.2\,\pi}$$



Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.