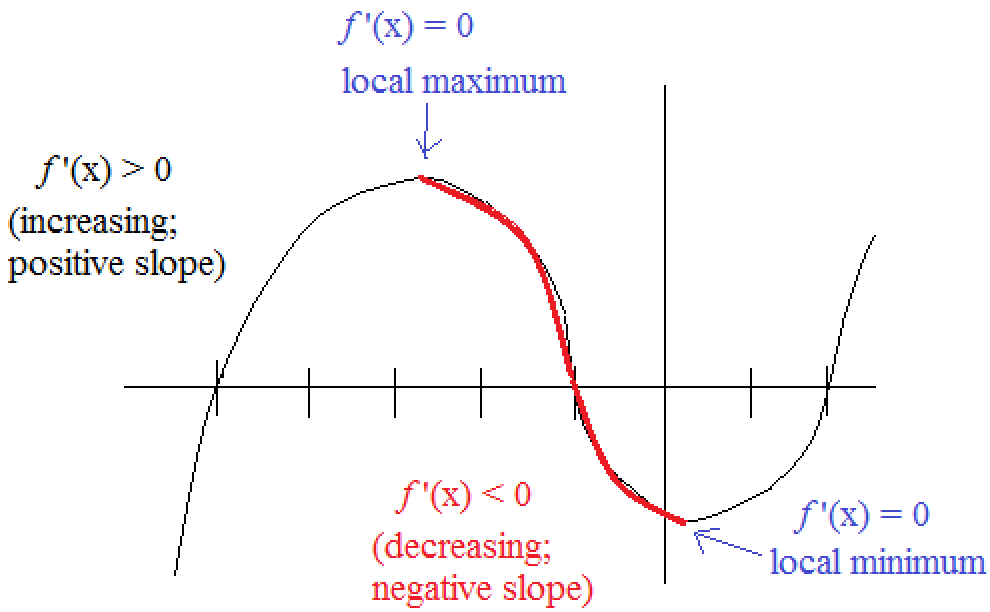
Sketching Graphs 1: Derivatives

Notes, examples, and practice test (with solutions)



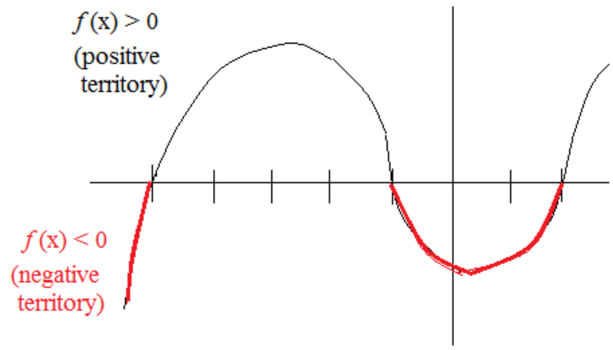
Topics include maximum/minimum, concavity, slope, velocity, acceleration, and more.

Derivatives and Graphs

The Function

$f(x)$: position of each x

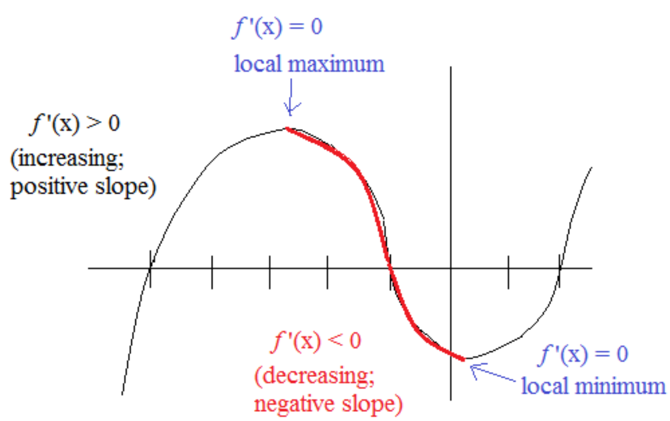
$f(x) > 0$	positive (above the x-axis)
$f(x) < 0$	negative (below the x-axis)
$f(x) = 0$	on the x-axis



The 1st derivative of the function

$f'(x)$: instantaneous rate of change (slope) at each x

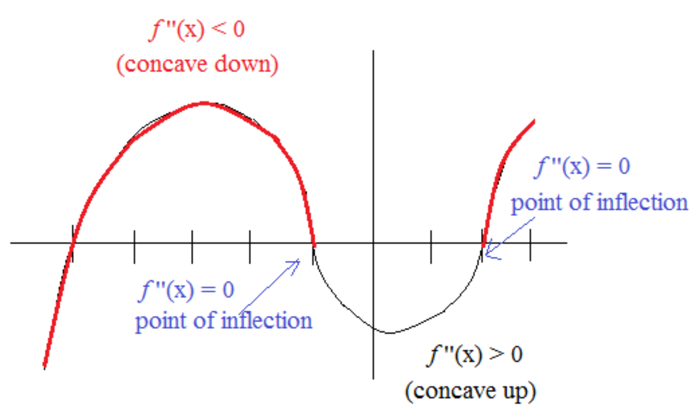
$f'(x) > 0$	increasing
$f'(x) < 0$	decreasing
$f'(x) = 0$	critical value (max/min)



The 2nd derivative of the function

$f''(x)$: acceleration (concavity) at each x

$f''(x) > 0$	concave up
$f''(x) < 0$	concave down
$f''(x) = 0$	point of inflection

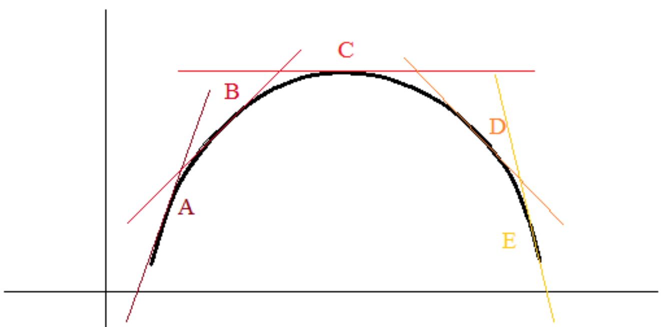


Derivatives and Graphs (continued)

Concavity and the 2nd derivative

The 1st derivative of a function describes the instantaneous rate of change of the function. (slope)

The 2nd derivative describes the instantaneous rate of change of the 1st derivative. (concavity)



Concave down:
The slope (1st derivative) decreases.

slope A = 2 1/2

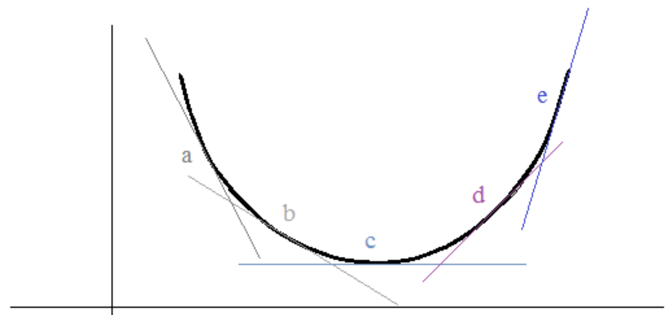
slope B = 1

slope C = 0 (maximum)

slope D = -1

slope E = -3

"The rate of change of the function is decreasing."
(Or, "the rate of change of the rate of change is negative!")



Concave up:
The slope (1st derivative) increases.

slope a = -2

slope b = -1/2

slope c = 0 (minimum)

slope d = 1

slope e = 3

"The rate of change of the function is increasing."
(Or, "the rate of change of the rate of change is positive!")

Maximum or minimum?

If $f'(x) = 0$, it is a critical value --- a maximum or a minimum.

How do you determine if it's a maximum or a minimum?

Look at the first derivative:

Pick a value on the left, and pick a value on the right....

If increasing on the left and decreasing on the right, then it's a maximum.
If decreasing on the left and increasing on the right, then it's a minimum.

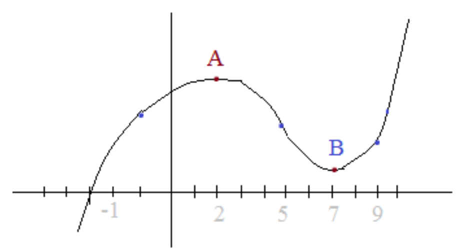
OR,

Look at the second derivative:

For x, where $f'(x) = 0$,

if $f''(x) < 0$, it is concave down; therefore, it's a maximum.

if $f''(x) > 0$, it is concave up; therefore, it's a minimum.



point A is a local maximum
$f'(2) = 0$ $f''(2) < 0$

(left) $f'(-1) > 0$ (increasing)
(right) $f'(5) < 0$ (decreasing)

point B is a local minimum
$f'(7) = 0$ $f''(7) > 0$

(left) $f'(5) < 0$ (decreasing)
(right) $f'(9) > 0$ (increasing)

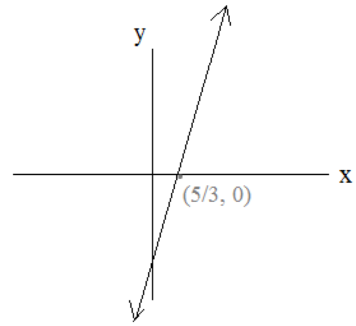
Derivatives and Graphs (continued)

Example 1: $y = 3x - 5$

The x-intercept is (5/3, 0). And, $y < 0$ for $(-\infty, 5/3)$ and $y > 0$ for $(5/3, \infty)$

first derivative: $y' = 3$ Since $3 > 0$, the slope is always increasing.

second derivative: $y'' = 0$ There is no concavity.



Example 2: $f(x) = x^2 - 10x + 16$

If we factor the equation, $(x - 2)(x - 8)$, we determine the x-intercepts are (2, 0) and (8, 0).
And, $f(x)$ is positive for the intervals $(-\infty, 2)$ and $(8, \infty)$
negative for the interval (2, 8)

$f'(x) = 2x - 10$

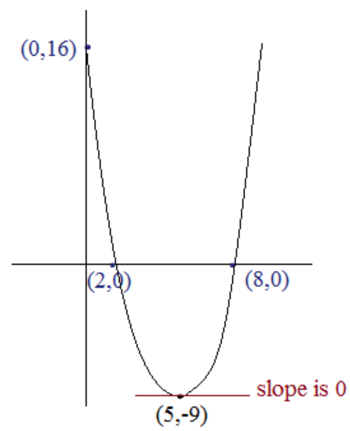
Set $f'(x) = 0$ to find critical values.

$$2x - 10 = 0$$
$$x = 5$$
$$f(5) = (5)^2 - 10(5) + 16 = -9$$ (5, -9) is relative minimum.

$2x - 10 < 0$ for all $x < 5$ function is decreasing on interval $(-\infty, 5)$

$2x - 10 > 0$ for all $x > 5$ function is increasing on interval $(5, \infty)$

$f''(x) = 2$ Since $2 > 0$, the function is concave up



Example 3: $g(x) = x^3 + 9x^2 + 24x - 2$

$g'(x) = 3x^2 + 18x + 24$

To find critical values, set first derivative equal to zero:

$$3(x^2 + 6x + 8) = 0$$
$$3(x + 2)(x + 4) = 0 \qquad x = -2, -4$$

$$(-2)^3 + 9(-2)^2 + 24(-2) - 2 = -22$$
$$(-4)^3 + 9(-4)^2 + 24(-4) - 2 = -18$$

critical points: (-2, -22) (-4, -18)

max or min? Check 2nd derivative.

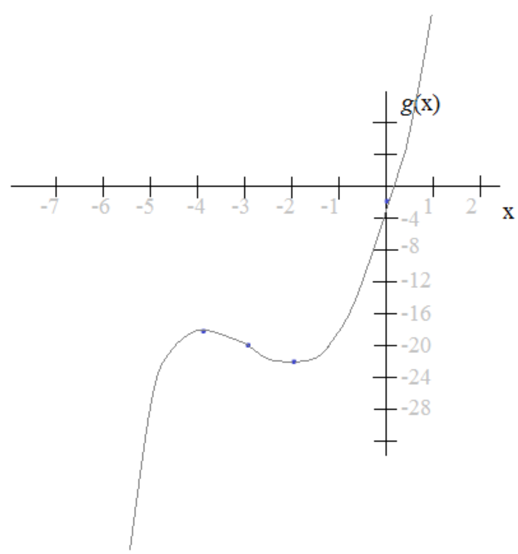
$g''(x) = 6x + 18$ $g''(-2) = 6$ (concave up; local minimum)

$g''(-4) = -6$ (concave down; local maximum)

To add detail to the sketch: y-intercept is (0, -2)

point of inflection: (-3, -20)

(because $g''(-3) = 0$)



Word Problem: Velocity and Acceleration (illustrating 1st and 2nd derivatives)

Suppose $h(t) = -16t^2 + 48t + 160$ represents the height of a ball (in feet) at a given time t seconds

a) If the ball was thrown from a balcony, how high is the balcony?

This is a position question; $h(0) = -16(0)^2 + 48(0) + 160 = 160$ feet

b) What is the initial velocity?

This is an instantaneous rate of change question; $h'(t) = -32t + 48$

since the initial velocity occurs at t = 0, the initial velocity is 48 feet/second

c) When does the ball reach maximum height? What is the maximum height?

To find a maximum, use the first derivative; $h'(t) = -32t + 48$

$h'(t) = 0$ will determine critical values

$-32t + 48 = 0 \qquad t = 3/2$

The ball reaches maximum height at 3/2 seconds.

Note: $h''(t) = -32$ Since $h''(t) < 0$, the entire function is concave down. (only a maximum could exist)

To find the maximum height, use the original function.

$h(3/2) = -16(3/2)^2 + 48(3/2) + 160 = -36 + 72 + 160 = 196$ feet

d) What is the acceleration of the ball?

Acceleration is the instantaneous rate of change of the velocity; use the 2nd derivative $h''(t) = -32$ feet/sec^2

e) When does the ball hit the ground?

This is a position question; find where $h(t) = 0$

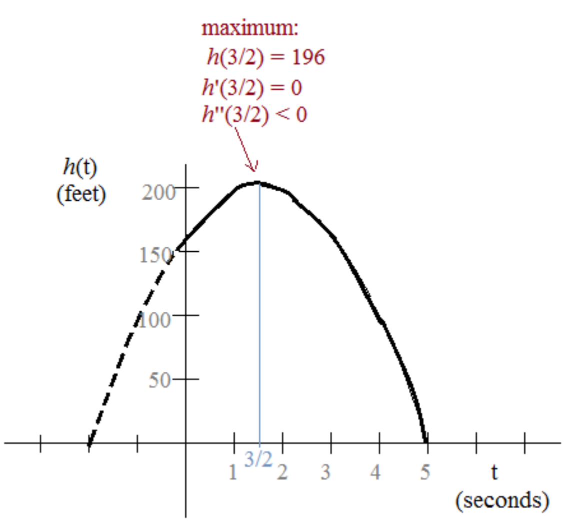
$-16t^2 + 48t + 160 = 0$

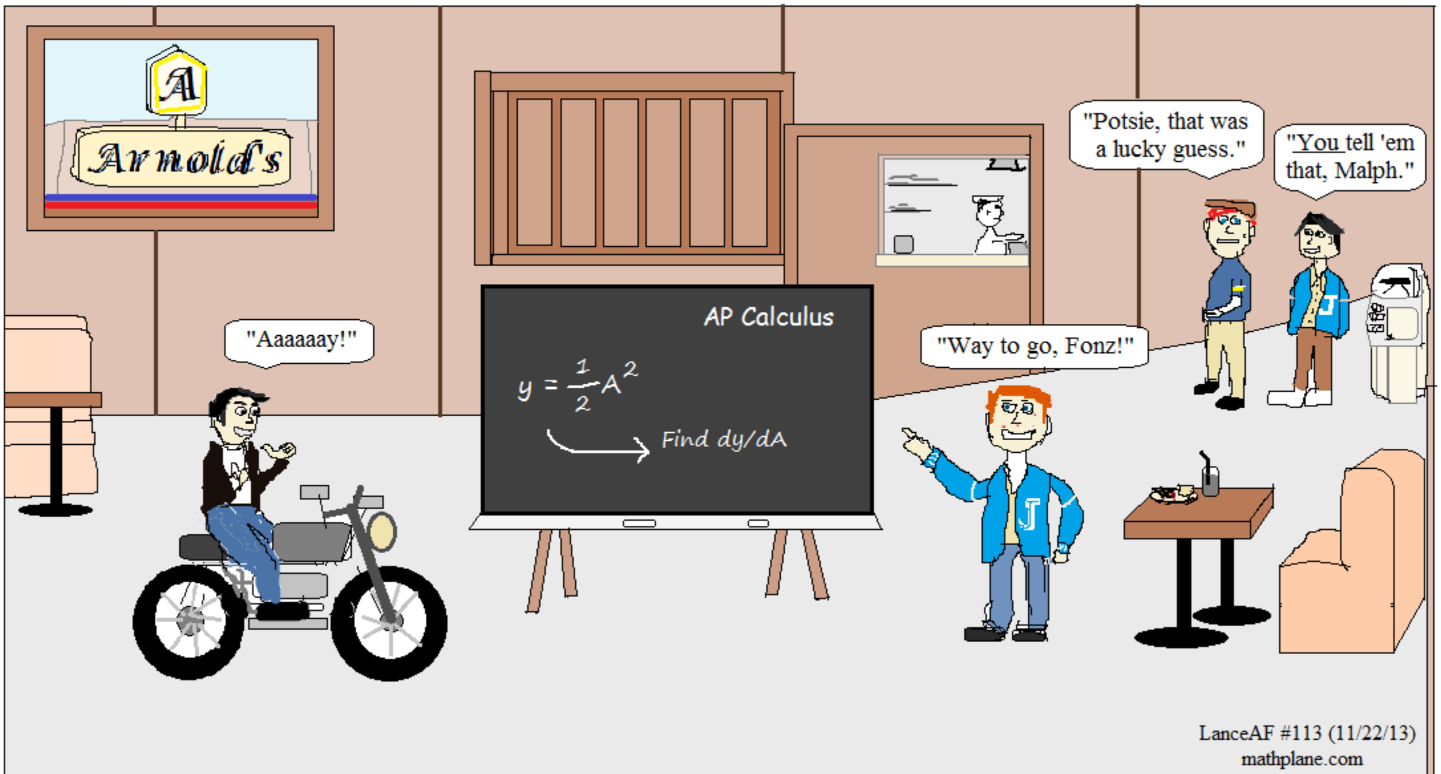
$-16(t^2 - 3t - 10) = 0$

$-16(t - 5)(t + 2) = 0$

$t = -2, 5$

The ball is on the ground at 5 seconds. (time cannot be -2!)





Happy Days

Despite Richie's help, Fonzie dropped out of Calculus.
(... although he did have some success with velocity and acceleration!)

Practice Quiz (and solutions) -→

Derivatives and graphs Quiz

I. Solve:

1) $f(x) = x^5 + 3x^3 - 4x + 9$

a) $f(-1) =$

b) $f'(2) =$

c) $f''(0) =$

2) $g(t) = -t^2 + 5t + 11$

a) $g(5) =$

b) $g'(1) =$

c) $g''(3) =$

II. Answer:

$f(x) = x^4 + 12x^3 - 20x^2 + 7$

a) What is the y-intercept?

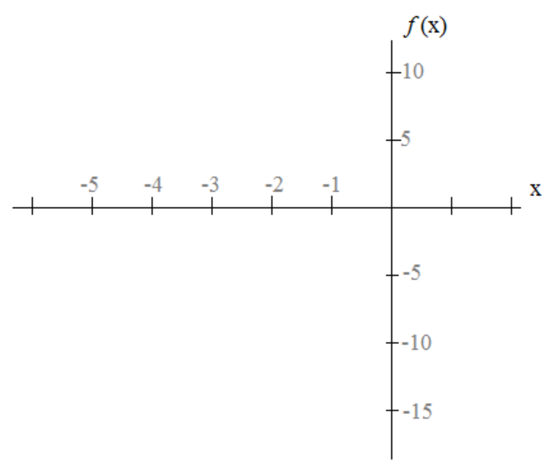
b) Identify any relative maximum(s).

c) Where are the points of inflection?

III. Find the first and second derivatives of the function $f(x) = x^3 + 6x^2 + 9x$

Identify the x-intercept(s), y-intercept, and any critical values. Describe the concavity.

Then, sketch a graph of the function.



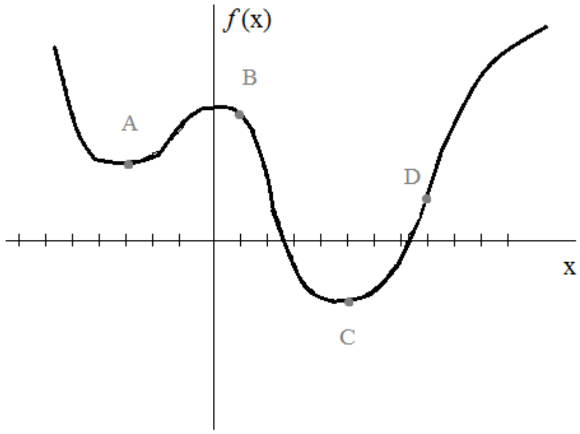
IV. For each point, determine whether the values are < > or = to zero.

A) $f(-3)$
$f'(-3)$
$f''(-3)$

B) $f(1)$
$f'(1)$

C) $f(5)$
$f''(5)$

D) $f'(8)$
$f''(8)$

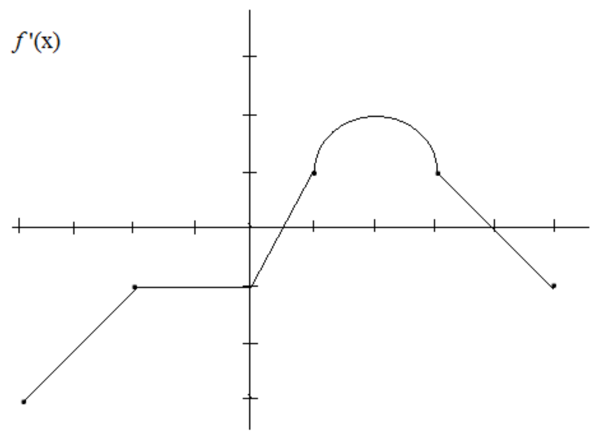


Challenge

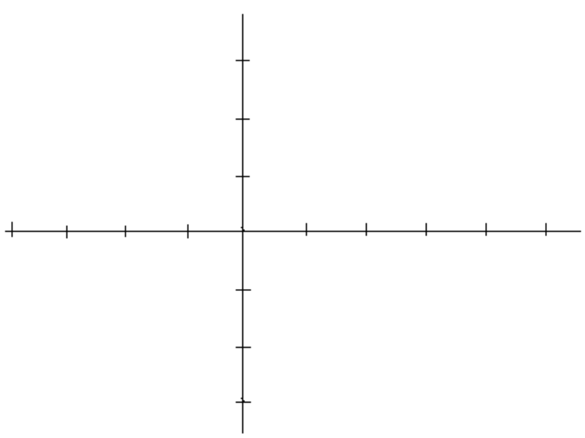
V. The following is the graph of $f'(x)$
(** It's the graph of the derivative of $f(x)$)

1) Sketch a graph of $f(x)$.

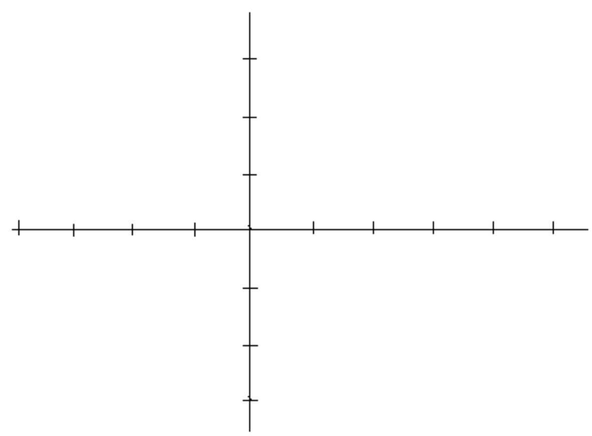
2) Sketch a graph of $f''(x)$.



1) $f(x)$



2) $f''(x)$



Derivatives and graphs Quiz

SOLUTIONS

I. Solve:

1) $f(x) = x^5 + 3x^3 - 4x + 9$

a) $f(-1) = (-1)^5 + 3(-1)^3 - 4(-1) + 9$
$-1 - 3 + 4 + 9 = \boxed{9}$

b) $f'(2) = $ $f'(x) = 5x^4 + 9x^2 - 4$
$f'(2) = 5(2)^4 + 9(2)^2 - 4 = \boxed{112}$

c) $f''(0) = $ $f''(x) = 20x^3 + 18x$
$f''(0) = 20(0)^3 + 18(0) = \boxed{0}$

2) $g(t) = -t^2 + 5t + 11$

a) $g(5) = -(5)^2 + 5(5) + 11 = -25 + 25 + 11 = \boxed{11}$

b) $g'(1) = $ $g'(t) = -2t + 5$
$g'(1) = -2(1) + 5 = \boxed{3}$

c) $g''(3) = $ $g''(t) = -2$
$g''(3) = \boxed{-2}$

II. Answer:

$f(x) = x^4 + 12x^3 - 20x^2 + 7$

a) What is the y-intercept? The point where the function crosses the y-axis: (0, ?)
$f(0) = (0)^4 + 12(0)^3 - 20(0)^2 + 7 = 7$ $\boxed{(0, 7)}$

b) Identify any relative maximum(s).
Relative max: any x, where $f'(x) = 0$ and $f''(x) < 0$
$\boxed{\text{(0, 7) is the only relative maximum}}$
(1, 0) and (-10, -3993) are relative minimums

$f'(x) = 4x^3 + 36x^2 - 40x$
$4x(x^2 + 9x - 10) = 0$
$4x(x + 10)(x - 1) = 0$
critical points @ x = 0, 1, -10

$f''(x) = 12x^2 + 72x - 40$
$f''(0) = -40$ concave down
$f''(1) = 44$ concave up
$f''(-10) = 440$ concave up

c) Where are the points of inflection?
point of inflection: second derivative equals 0

$$\boxed{x = \frac{-9 \pm \sqrt{111}}{3}}$$

$f''(x) = 12x^2 + 72x - 40$
$4(3x^2 + 18x - 10) = 0$

(quadratic formula) $\dfrac{-18 \pm \sqrt{324 + 120}}{6}$

$\dfrac{-9 \pm \sqrt{111}}{3}$

III. Find the first and second derivatives of the function $f(x) = x^3 + 6x^2 + 9x$
Identify the x-intercept(s), y-intercept, and any critical values. Describe the concavity.
Then, sketch a graph of the function.

$f'(x) = 3x^2 + 12x + 9$

$f''(x) = 6x + 12$

x-intercepts: $f(x) = 0$

$\boxed{(0, 0) \quad (-3, 0)}$

$x^3 + 6x^2 + 9x = 0$
$x(x^2 + 6x + 9) = 0$
$x(x + 3)(x + 3) = 0$
$x = 0, -3$

y-intercept: $f(0) = $
$(0)^3 + 6(0)^2 + 9(0) = 0$

$\boxed{(0, 0)}$

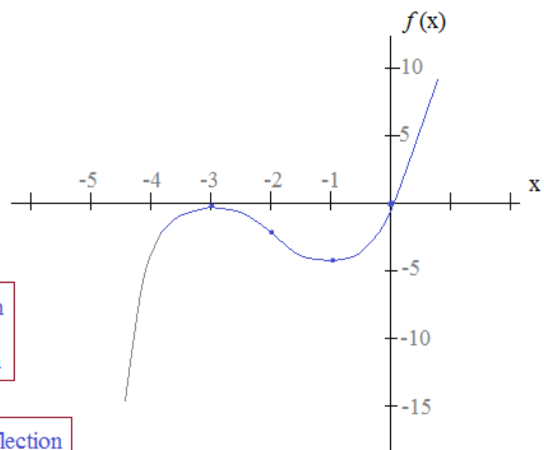
critical values: $f'(x) = 3x^2 + 12x + 9 = 0$
$3(x^2 + 4x + 3) = 0$
$3(x + 1)(x + 3) = 0$
$x = -1$ and -3

$\boxed{\text{(-1, -4) is a minimum; (-3, 0) is a maximum}}$

points of inflection: $f''(x) = 6x + 12 = 0$
$6(x + 2) = 0$
$x = -2$

$\boxed{\text{(-2, -2) is point of inflection}}$

$\boxed{\text{concave down } x < -2; \text{ concave up } x > -2}$



IV. For each point, determine whether the values are < > or = to zero.

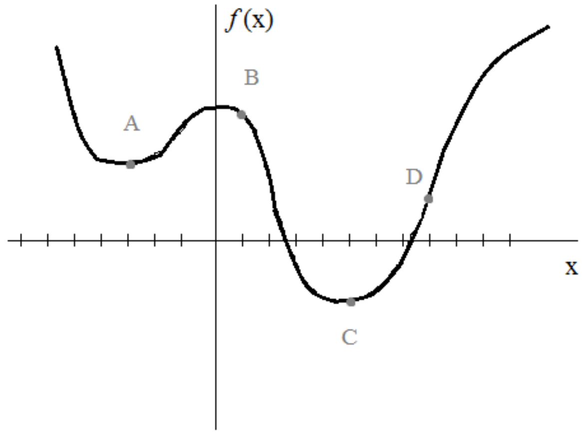
SOLUTIONS

A) $f(-3)$ > 0
$f'(-3)$ $= 0$ (local minimum)
$f''(-3)$ > 0 (concave up)

B) $f(1)$ > 0
$f'(1)$ < 0 (negative slope)

C) $f(5)$ < 0
$f''(5)$ > 0 (concave up)

D) $f'(8)$ > 0 (positive slope)
$f''(8)$ $= 0$ (point of inflection)



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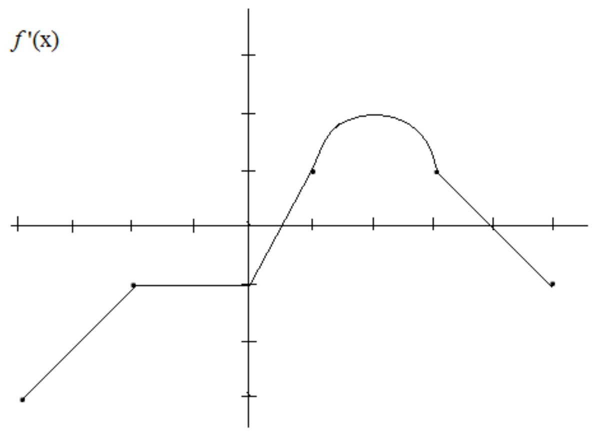
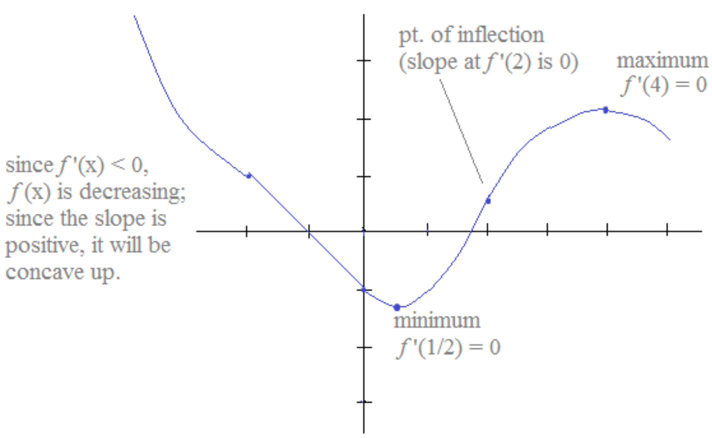
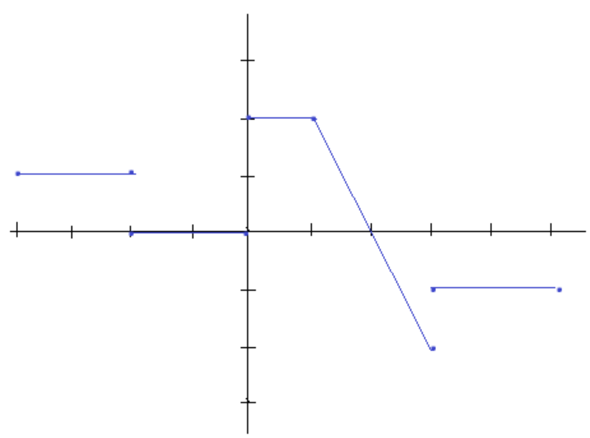


1) $f(x)$



2) $f''(x)$



$f''(x)$ is the derivative of $f'(x)$, so the sketch describes the instantaneous rates of change (slopes) of each x in $f'(x)$