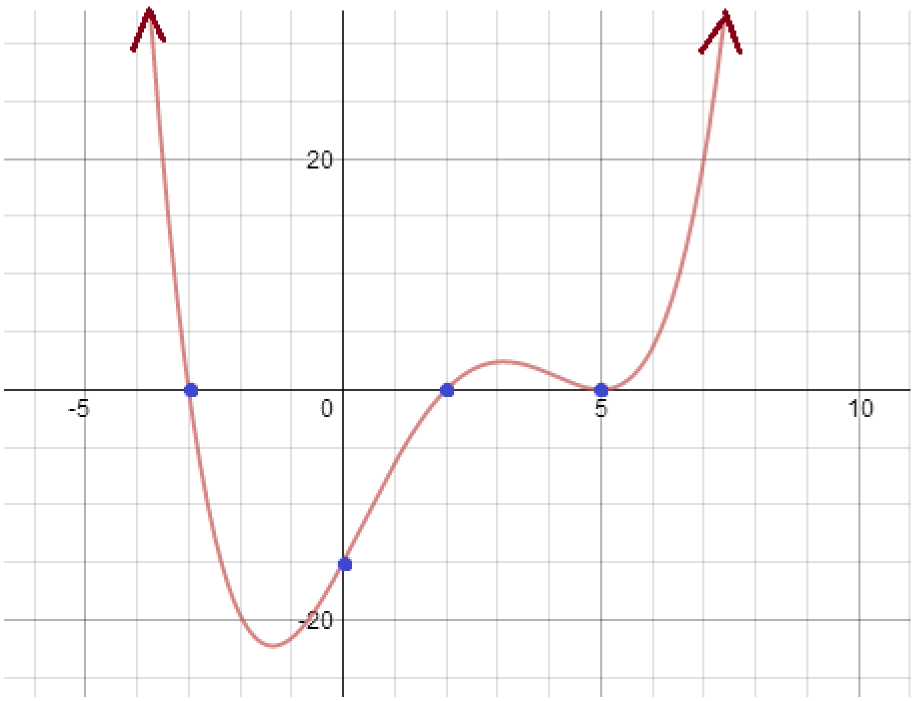
Polynomials 2: Factors, Roots, and Theorems

Examples, Notes, and Practice Tests (with Solutions)



Topics include rational root, factoring, conjugates, graphing, synthetic division, long division, sum and product rules, and more.

Example: Factor and Sketch the Polynomial

$g(x) = x^4 + 8x^3 + 15x^2 - 4x - 20$

(There is no greatest common factor (GCF))

Step 1: Use Rational Root Theorem to find factor

'p': factors of 20: 1, 2, 4, 5, 10, 20
'q': factors of 1: 1

Possible rational roots: $\pm\frac{'p'}{'q'}$ $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$

Step 2: Use Remainder/Factor Theorem to select a root

Test (-1): $g(-1) = (-1)^4 + 8(-1)^3 + 15(-1)^2 - 4(-1) - 20$

$= -8$ Since $g(-1) = -8$, the remainder of $g(x) \div (x + 1)$ is 8...

It is NOT a root (factor)

Test (1): $g(1) = (1)^4 + 8(1)^3 + 15(1)^2 - 4(1) - 20$

$= 0$ Since $g(1) = 0$, the remainder of $g(x) \div (x - 1)$ is 0

It is a root (factor)

$(x - 1)$

Step 3: Use Synthetic Division to reduce the polynomial by 1 degree

1	1	8	15	-4	-20
		1	9	24	20
	1	9	24	20	0

$x^3 + 9x^2 + 24x + 20$ $h(x)$

Repeat Steps 1, 2, and 3 to find the next root:

$x^3 + 9x^2 + 24x + 20$

'p': factors of 20: 1, 2, 4, 5, 10, 20
'q': factors of 1: 1

Possible rational roots: $\pm\frac{'p'}{'q'}$ ~~$\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$~~

~~-1~~, -2, -4, -5, -10, -20

Note: Since each term in the polynomial has a positive coefficient, the remaining rational roots will NOT be positive!
(i.e. using the remainder theorem, there is no way $h(x) = 0$ if x is a positive number...)

Since -1 did not work before, we'll test -2

$h(-2) = (-2)^3 + 9(-2)^2 + 24(-2) + 20$

$= 0$

$(x + 2)$

-2	1	9	24	20
		-2	-14	-20
	1	7	10	0

$x^2 + 7x + 10x$

Step 4: Factor the quadratic..

$x^2 + 7x + 10x = (x + 2)(x + 5)$

Note: the discriminant $b^2 - 4ac = 49 - 40 = 9$
since $9 > 0$, the solutions/zeros are REAL
and, since 9 is a perfect square, they are RATIONAL

$(x + 2)(x + 5)$

Step 5: Using the factors and equation, sketch the curve

$g(x) = x^4 + 8x^3 + 15x^2 - 4x - 20$

or $g(x) = (x + 2)^2(x - 1)(x + 5)$

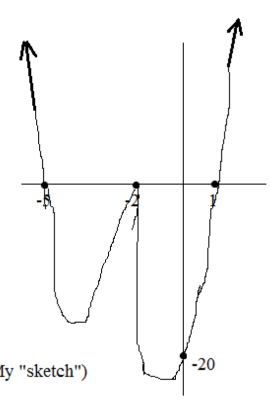
x-intercepts are the zeros: -2, -2, 1, -5 ---> (-2, 0) (1, 0) and (-5, 0)
y-intercept is (0, -20)

end behavior: The degree is 4, so the ends go in the same direction..
The lead coefficient is positive, so the curve goes up

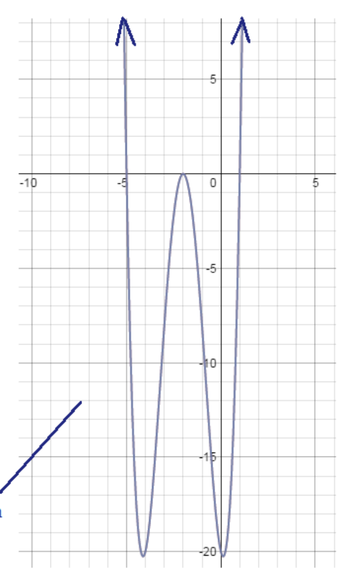
as $x \to \infty$, $g(x) \to \infty$

$x \to -\infty$, $g(x) \to \infty$

since the zero (-2) has a multiplicity of 2
(i.e. $(x + 2)(x + 2)$), there will be a "bounce" at -2



(My "sketch")



The graphed function

Random Polynomial Notes & Examples

Clarifying the polynomial terms:

Example: $y = x^4 - 3x^3 - 13x^2 + 15x$

zeros 1, -3, 5, 0

intercepts (1, 0) (-3, 0) (5, 0) (0, 0) or the origin

"roots" 1, -3, 5, 0

factors $(x-1)\ (x+3)\ (x)\ (x-5)$

Searching for rational roots: A Shortcut

Example: Find the roots of the polynomial

$f(x) = x^3 + 12x^2 + 21x + 10$

NOTE: When considering the p's and q's -- the possible rational roots -- we can eliminate all the positive possiblities!

There is no way that $f(x) = 0$ if x is positive....

Since the polynomial cannot be factored by grouping, we'll check for rational roots...

"p's" (factors of the constant): 1, 2, 5, 10
"q's" (factors of the lead term): 1

According to rational root theorem, possible rational roots include 1, 2, 5, 10, -1, -2, -5, -10

Which possible rational root shall we check first?

Since we can eliminate all the positive numbers, we'll start with -1:

$f(-1) = (-1)^3 + 12(-1)^2 + 21(-1) + 10 = 0$

(according to remainder/factor theorems, -1 is a root)

$(x + 1)$

-1	1	12	21	10
		-1	-11	-10
	1	11	10	0

$x^2 + 11x + 10$

$(x + 10)(x + 1)$

$f(x) = (x + 1)^2\ (x + 10)$

Sum and Product Rules of Roots:

Example: Write a quadratic equation with zeros $(3 + 5i)$ and $(3 - 5i)$

Method 1: Using sum and product rule of roots

SUM of the roots: $(3 + 5i) + (3 - 5i) = 6$

PRODUCT of the roots: $(3 + 5i)(3 - 5i) =$

$9 + 15i - 15i - 25i^2$

$9 - (-25) = 34$

$\frac{\text{Sum of roots}}{A} = -B$

$\frac{\text{Product of roots}}{A} = C$

$1x^2 - 6x + 34$

Method 2: FOIL and binomials

$(x - (3 + 5i))(x - (3 - 5i))$

$(x - 3 - 5i)(x - 3 + 5i)$

$x\ (x - 3 + 5i)$ $\quad x^2 - 3x + 5ix$

$-3\ (x - 3 + 5i)$ $\quad -3x \quad + 9 - 15i$

$-5i\ (x - 3 + 5i)$ $\quad -5ix \quad + 15i - 25i^2$

$x^2 - 6x \quad + 9 \quad + 25$

$x^2 - 6x + 34$

Then, what is the cubic with zero -1

$(x + 1)(\ x^2 - 6x + 34\ )$

zero is -1,
so factor is (x + 1)

$x\ (x^2 - 6x + 34)$ $\quad x^3 - 6x^2 + 34x$

$1\ (x^2 - 6x + 34)$ $\quad x^2 - 6x + 34$

$x^3 - 5x^2 + 28x + 34$

Example: Given the polynomial $g(x) = x^7 - 3x^2 - 1$

What is the remainder of $\frac{g(x)}{(x-3)}$?

Evaluate with synthetic division... Then, check with remainder theorem....

(synthetic division)

3	1	0	0	0	0	-3	0	-1
		3	9	27	81	243	720	2160
	1	3	9	27	81	240	720	**2159**

(remainder theorem)

$g(3) = (3)^7 - 3(3)^2 - 1$

$= 2187 - 27 - 1$

$= \boxed{2159}$

Conjugate Root Theorem

Since $3i$ is a root, then $-3i$ must be a root, too...

$(x + 3i)(x - 3i) = x^2 + 3ix - 3ix - 9i^2$

$= \boxed{x^2 + 9}$

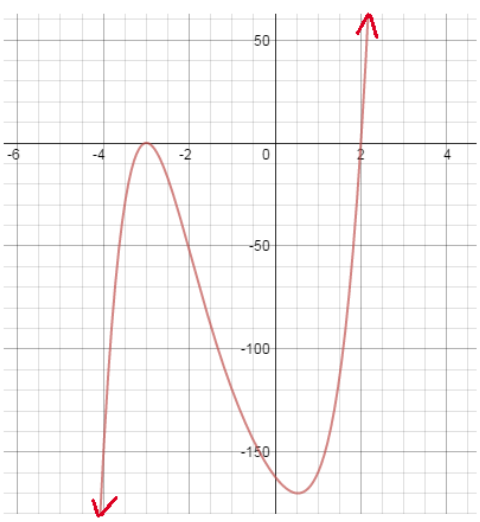
Example: $x^5 + 4x^4 + 6x^3 + 18x^2 - 27x - 162$

If $3i$ is a zero, find the other zeros...

Then, write the polynomial in factored form...

Polynomial Long Division

$$\begin{array}{r} x^3 + 4x^2 - 3x - 18 \\ (x^2+9)\overline{)\, x^5 + 4x^4 + 6x^3 + 18x^2 - 27x - 162} \\ -\underline{(x^5 \qquad + 9x^3) \qquad\qquad\qquad} \\ 4x^4 - 3x^3 + 18x^2 \qquad\qquad \\ -\underline{(4x^4 \qquad + 36x^2) \qquad\quad} \\ -3x^3 - 18x^2 - 27x \qquad \\ -\underline{(-3x^3 \qquad\quad - 27x) \qquad} \\ -18x^2 \qquad - 162 \\ -\underline{(-18x^2 \qquad - 162)} \\ 0 \end{array}$$



Rational Root Theorem

Possible rational roots: "p": 1, 2, 3, 6, 9, 18
"q": 1

1, -1, 2, -2, 3, -3, 6, -6, 9, -9, 18, -18

Remainder/Factor Theorem

If x = 1, $(1)^3 + 4(1)^2 - 3(1) - 18 = -16$ NOT a factor

If x = -1, $(-1)^3 + 4(-1)^2 - 3(-1) - 18 = -12$ NOT a factor

If x = 2, $(2)^3 + 4(2)^2 - 3(2) - 18 = 0$ FACTOR!! $\boxed{(x - 2)}$

Synthetic Division

2	1	4	-3	-18
		2	12	18
	1	6	9	**0**

$\rightarrow x^2 + 6x + 9$

Factor a quadratic

$x^2 + 6x + 9 = \boxed{(x + 3)(x + 3)}$

Zeros are $3i$, $-3i$, -3, and 2

Polynomial in factored form: $(x^2 + 9)(x - 2)(x + 3)^2$

Example: Factor: $8x^5 - 25y^3 + 80x^4 - x^2y^3 + 200x^3 - 10xy^3$

$8x^5 + 80x^4 + 200x^3 - x^2y^3 - 10xy^3 - 25y^3$ rearrange

$8x^3(x^2 + 10x + 25) - y^3(x^2 + 10x + 25)$ group and greatest common factors (GCF)

$(8x^3 - y^3)\cdot(x^2 + 10x + 25)$ regroup

$(2x - y)(4x^2 + 2xy + y^2)\ (x + 5)(x + 5)$ different of cubes and factoring

$(2x - y)(4x^2 + 2xy + y^2)(x + 5)^2$

Example: Factor: $2x^4 + x^3 - 8x^2 - x + 6$

$x^3 - x + 2x^4 - 8x^2 + 6$ rearrange

$x(x^2 - 1) + 2(x^4 - 4x^2 + 3)$ greatest common factor

$x(x^2 - 1) + 2(x^2 - 1)(x^2 - 3)$ factor the trinomial

$(x^2 - 1)(x + 2(x^2 - 3))$ regroup (using greatest common factor)

$(x^2 - 1)(2x^2 + x - 6)$

$(x + 1)(x - 1)(2x^2 + x - 6)$ difference of squares

"A different kind of grouping"

Example: Factor $x^2 - 4x + 4 - 100y^2$

The polynomial has 4 terms, and it can be grouped...
The first 3 terms separated from the fourth:

$x^2 - 4x + 4 \quad - 100y^2$

$(x - 2)(x - 2) \quad - 100y^2$

$(x - 2)^2 - 100y^2$ then, difference of squares

$(x - 2 + 10y)(x - 2 - 10y)$

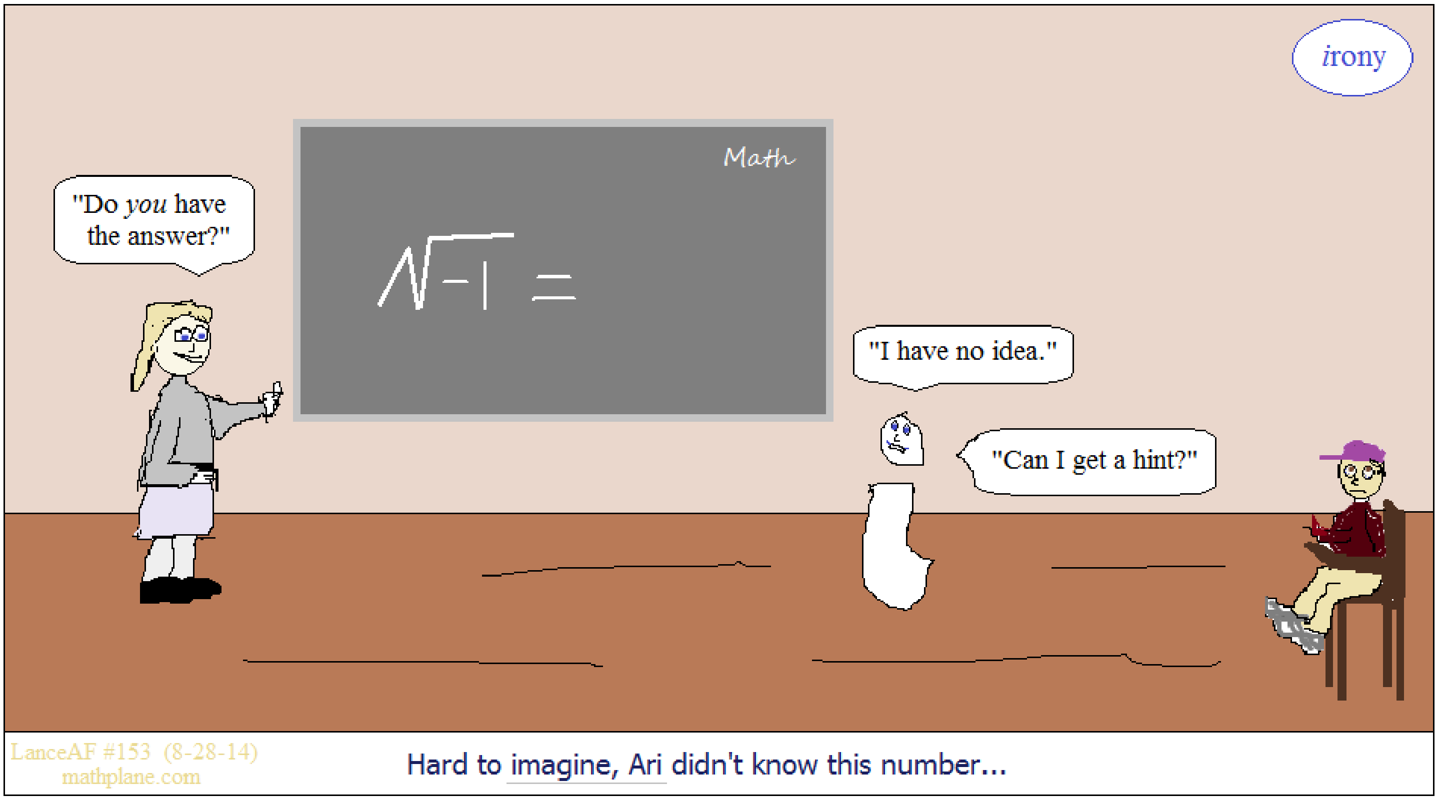
Example: Factor $a^2 + 8a + 16 - b^2$

$a^2 + 8a + 16 \quad - b^2$

$(a + 4)(a + 4) \quad - b^2$

$(a + 4)^2 - b^2$

$(a + 4 - b)(a + 4 + b)$



Practice Exercises-→

Factoring Quiz (Advanced)

Factor or solve each equation.... ***Which of the following is not a polynomial?

1) $x^6 - 7x^3 + 6$

2) $9a^4 b - 12a^2 b^4 + 4b^7$

3) $x^6 - 2x^5 + x^4 - x^2 + 2x - 1$

4) $x^2 - y^2 - 10y - 25$

5) $25 - x^2 - 4xy - 4y^2$

6) $2x^3 - 3ax^2 - 18x + 27a$

7) $20x^3 - 8x^2 - 35x + 14 = 0$

8) $x^2(x - 2) + x(x - 2)^2 = 0$

9) $27x^3 + 21x^2 - 14x - 8$

10) $(a^2 + 2a)^2 - 2(a^2 + 2a) - 3$

11) $x^{\frac{3}{2}} + 5x^{\frac{-1}{2}} - 6x^{\frac{1}{2}} = 0$

12) $(x^2 + 1)^2 - 7(x^2 + 1) + 10$

Solve with long division. Then, check answer (applying the remainder/factor theorems).

1) $x^4 + 8x^3 + 5x^2 - 38x + 24 \div (x + 1)$

2) $x^3 - 10x^2 + 6 \div (x - 2)$

3) $x^4 - x^2 - 3 \div (x - \sqrt{2})$

Solve with long division.

Polynomial Long Division

4) $x^5 + 4x^4 + 6x^3 + 18x^2 - 27x - 162 \div (x^2 + 9)$

5) $x^2 - 4x - 2 \div [x - (3 + 4i)]$

6) $x^6 - 4x^3 + 6x^2 + 10 \div (2x^2 + 5)$

1) Find the factors of $x^4 + x^3 - x^2 + x - 2$

2) Find a 4th degree polynomial with zeros $3i$ and -1 , -1 has a multiplicity of 2.

3) A) Write a quadratic equation in standard form whose solutions are $(2 + 5i)$ and $(2 - 5i)$

B) Now, write a cubic equation whose solutions (zeros) are -1, $2 + 5i$, and $2 - 5i$

C) And, finally, write a quartic equation in factored form whose solutions (zeros) are -1, $2 + 5i$, and $2 - 5i$

4) Using synthetic division, determine the following:

A) $x^3 + 7x - 1 \div (x - 2)$

B) $3x^3 - 2x^2 + 5x + 1 \div (3x + 1)$

5) Why can't +1 be a possible root of

$f(x) = 3x^4 + x^3 + 9x^2 + 6x + 15$?

(in other words, why is $(x - 1)$ NOT a factor?)

6) $g(x) = 2x^3 - 10x^2 + 200x - 1000$

Consider the possible rational roots.
Which ones should you omit?
Which should you test first?
Explain.

7) Find the zeroes: $f(x) = x^6 + 6x^5 + 15x^4 + 36x^3 + 54x^2$

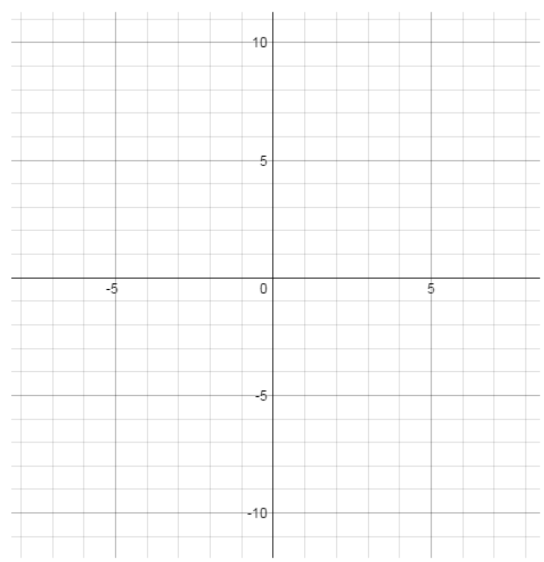
8) Let $g(x) = 2x^3 - 5x^2 - 4x + 3$

Find the *possible* rational zeros of g

Determine the complete factorization

What are the x-intercepts? The y-intercept?

Sketch the graph

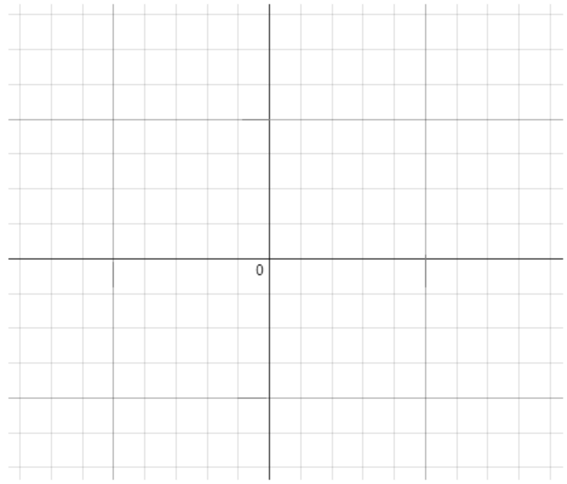


Polynomials Concepts: Graphing

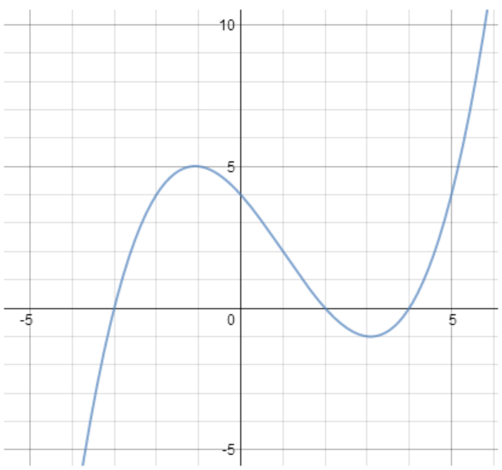
Sketch the following polynomial:

$$y = \frac{1}{10}(x + 3)(x - 2)(x - 5)^2$$

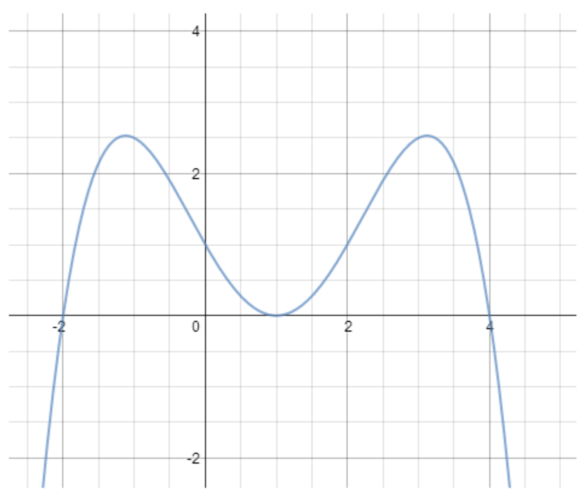
Label the intercepts...



Write the equation of the polynomial in the following graph:



Write the equation of the polynomial in the following graph:



Polynomials: Concepts and Synthetic Division

The following are results of synthetic division from the function Q(x) and selected linear binomials....

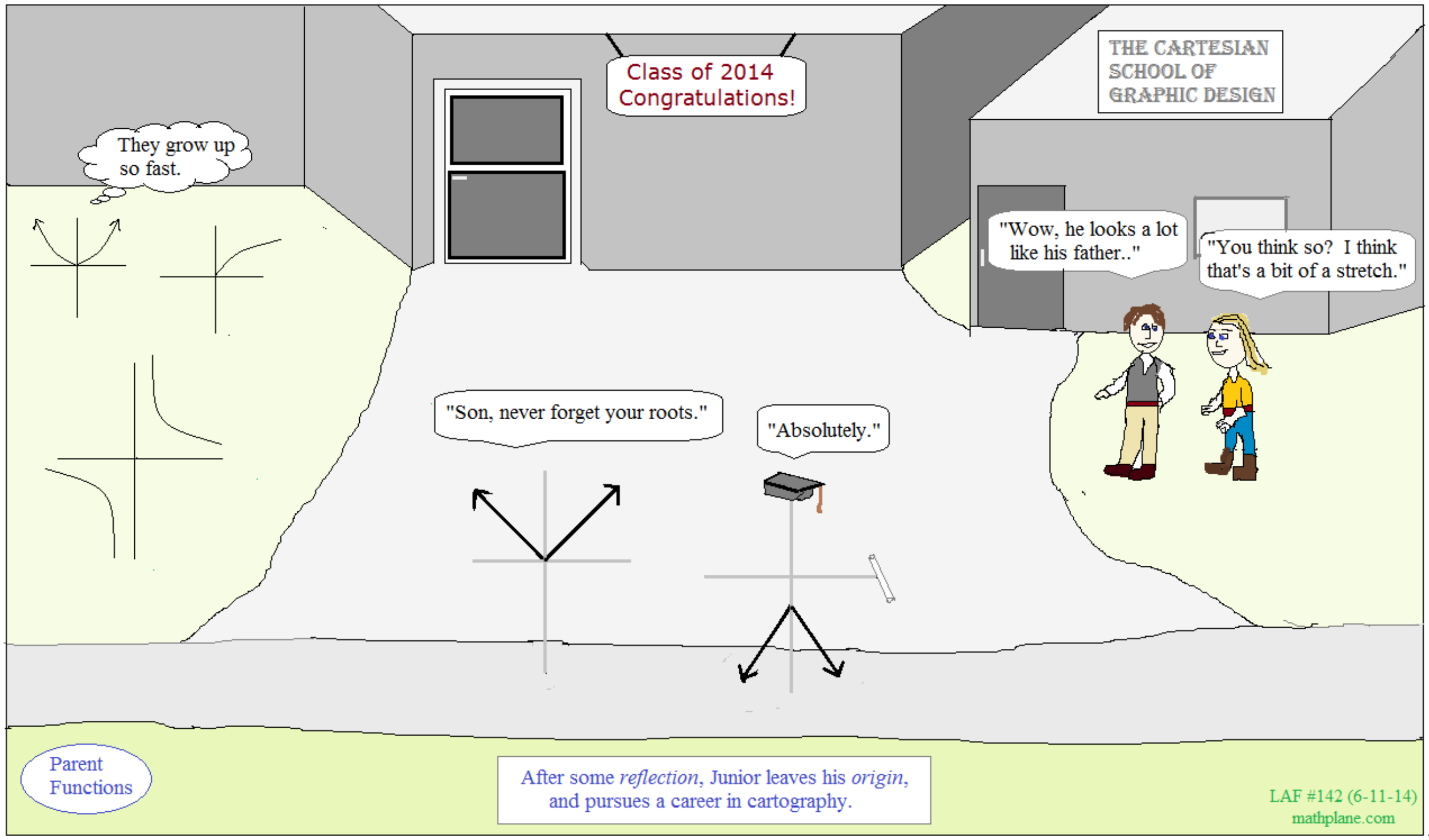
Divisor					Remainder
−5	1	-8	41	-220	1080
−4	1	-7	29	-131	504
−3	1	-6	19	-72	196
−2	1	-5	11	-37	54
−1	1	-4	5	-20	0
1	1	-2	-1	-16	-36
2	1	-1	-1	-17	-54
3	1	0	1	-12	-56
4	1	1	5	5	0
5	1	2	11	40	180

1) What is Q(3)?

2) Find $\frac{Q(x)}{x+2}$

3) Find the equation of the polynomial Q(x) in *standard form.*

4) Write the equation of the polynomial as a product of linear factors (or factored form)


Class of 2014
Congratulations!
THE CARTESIAN
SCHOOL OF
GRAPHIC DESIGN
They grow up
so fast.
"Wow, he looks a lot
like his father.."
"You think so? I think
that's a bit of a stretch."
"Son, never forget your roots."
"Absolutely."
Parent
Functions
After some *reflection*, Junior leaves his *origin*,
and pursues a career in cartography.
LAF #142 (6-11-14)
mathplane.com

Solutions -→

Factoring Quiz (Advanced) — Factor or solve each equation.... ***Which of the following is not a polynomial? — SOLUTIONS

1) $x^6 - 7x^3 + 6$

$(x^3 - 1)(x^3 - 6)$

$(x - 1)(x^2 + x + 1)(x^6 - 6)$

2) $9a^4 b - 12a^2 b^4 + 4b^7$

GCF $b \cdot [9a^4 - 12a^2 b^3 + 4b^6]$

$b \cdot (3a^2 - 2b^3)(3a^2 - 2b^3)$

$b(3a^2 - 2b^3)^2$

3) $x^6 - 2x^5 + x^4 - x^2 + 2x - 1$

$x^4(x^2 - 2x + 1) - 1(x^2 - 2x + 1)$

$(x^4 - 1)(x^2 - 2x + 1)$

$(x^2 - 1)(x^2 + 1)(x - 1)(x - 1)$

$(x + 1)(x - 1)(x^2 + 1)(x - 1)^2$

$(x + 1)(x^2 + 1)(x - 1)^3$

4) $x^2 - y^2 - 10y - 25$

$x^2 - 1 \cdot (y^2 + 10y + 25)$

$x^2 - (y + 5)(y + 5)$

$x^2 - (y + 5)^2$

$(x + y + 5)(x - y - 5)$

5) $25 - x^2 - 4xy - 4y^2$

$25 - (x^2 + 4xy + 4y^2)$

$25 - (x + 2y)(x + 2y)$

$25 - (x + 2y)^2$

$[5 - (x + 2y)] \cdot [5 + (x + 2y)]$

$(5 - x - 2y)(5 + x + 2y)$

6) $2x^3 - 3ax^2 - 18x + 27a$

$x^2(2x - 3a) - 9(2x - 3a)$

$(2x - 3a)(x^2 - 9)$

$(2x - 3a)(x + 3)(x - 3)$

7) $20x^3 - 8x^2 - 35x + 14 = 0$

Group the 1st and 3rd terms;
Group the 2nd and 4th terms.
Then, factor (GCF)

$5x(4x^2 - 7) + (-2)(4x^2 - 7) = 0$

Regroup: $(5x - 2)(4x^2 - 7) = 0$

Solve: $x = 2/5$ or $x = \pm\sqrt{\frac{7}{4}}$

8) $x^2(x - 2) + x(x - 2)^2 = 0$

GCF $x(x - 2) \cdot [x + (x - 2)] = 0$

$x(x - 2)(2x - 2) = 0$

$x = 0, 1, 2$

9) $27x^3 + 21x^2 - 14x - 8$

Group 1st and 4th terms;
Group 2nd and 3rd terms..

$27x^3 - 8 + 21x^2 - 14x$

$27x^3 - 8 + 7x(3x - 2)$ GCF

$(3x - 2)(9x^2 + 6x + 4) - 7x(3x - 2)$ Difference of Cubes

$(3x - 2)[(9x^2 + 6x + 4) - 7x]$ ReGroup

$(3x - 2)(9x^2 - x + 4)$

10) $(a^2 + 2a)^2 - 2(a^2 + 2a) - 3$

Let $B = (a^2 + 2a)$

$B^2 - 2B - 3$

$(B - 3)(B + 1)$

$(a^2 + 2a - 3)(a^2 + 2a + 1)$

$(a - 1)(a + 3)(a + 1)(a + 1)$

11) $x^{\frac{3}{2}} + 5x^{\frac{-1}{2}} - 6x^{\frac{1}{2}} = 0$

$x^{\frac{-1}{2}}(x^2 + 5 - 6x) = 0$

$\frac{x^2 + 5 - 6x}{\sqrt{x}} = 0$

$\frac{(x - 1)(x - 5)}{\sqrt{x}} = 0$

$x = 1, 5$

NOT a polynomial...
(fractional and negative exponents)

12) $(x^2 + 1)^2 - 7(x^2 + 1) + 10$

Let $U = x^2 + 1$

$U^2 - 7U + 10$

$(U - 2)(U - 5)$

$(x^2 + 1 - 2)(x^2 + 1 - 5)$

$(x^2 - 1)(x^2 - 4)$

$(x + 1)(x - 1)(x + 2)(x - 2)$

Solve with long division. Then, check answer (applying the remainder/factor theorems). SOLUTIONS

1) $x^4 + 8x^3 + 5x^2 - 38x + 24 \div (x + 1)$

$$
\begin{array}{r|l}
 & x^3 + 7x^2 - 2x - 36 + \frac{60}{x+1} \\
x + 1 & x^4 + 8x^3 + 5x^2 - 38x + 24 \\
 & -(x^4 + x^3) \\
 & 7x^3 + 5x^2 \\
 & -(7x^3 + 7x^2) \\
 & -2x^2 - 38x \\
 & -(-2x^2 - 2x) \\
 & -36x + 24 \\
 & -(-36x - 36) \\
 & 60
\end{array}
$$

(using the remainder theorem)

$f(-1) = (-1)^4 + 8(-1)^3 + 5(-1)^2 - 38(-1) + 24 =$

$1 + -8 + 5 + 38 + 24 = 60$ ✓

also, synthetic division:

−1	1	8	5	-38	24
		-1	-7	2	36
	1	7	-2	-36	60

$x^3 + 7x^2 - 2x - 36 + \frac{60}{x+1}$

2) $x^3 - 10x^2 + 6 \div (x - 2)$

$$
\begin{array}{r|l}
 & x^2 - 8x - 16 - \frac{26}{x-2} \\
x - 2 & x^3 - 10x^2 + 0x + 6 \\
 & -(x^3 - 2x^2) \\
 & -8x^2 + 0x \\
 & -(-8x^2 + 16x) \\
 & -16x + 6 \\
 & -(-16x + 32) \\
 & -26
\end{array}
$$

(using the remainder theorem)

$f(2) = (2)^3 - 10(2)^2 + 6 =$

$8 - 40 + 6 = -26$ ✓

also, synthetic division:

2	1	-10	0	6
		2	-16	-32
	1	-8	-16	-26

$x^2 - 8x - 16 - \frac{26}{x-2}$

3) $x^4 - x^2 - 3 \div (x - \sqrt{2})$

$$
\begin{array}{r|l}
 & x^3 + \sqrt{2}x^2 + x + \sqrt{2} + \frac{-1}{x-\sqrt{2}} \\
x - \sqrt{2} & x^4 + 0x^3 - x^2 + 0x - 3 \\
 & -(x^4 - \sqrt{2}x^3) \\
 & \sqrt{2}x^3 - x^2 \\
 & -(\sqrt{2}x^3 - 2x^2) \\
 & x^2 + 0x \\
 & -(x^2 - \sqrt{2}x) \\
 & \sqrt{2}x - 3 \\
 & -(\sqrt{2}x - 2) \\
 & -1
\end{array}
$$

(using the remainder theorem)

$f(\sqrt{2}) = (\sqrt{2})^4 - (\sqrt{2})^2 - 3 =$

$4 - 2 - 3 = -1$ ✓

also, synthetic division:

$\sqrt{2}$	1	0	-1	0	-3
		$\sqrt{2}$	2	$\sqrt{2}$	2
	1	$\sqrt{2}$	1	$\sqrt{2}$	-1

$x^3 + \sqrt{2}x^2 + x + \sqrt{2} + \frac{-1}{x-\sqrt{2}}$

Solve with long division.

SOLUTIONS

Polynomial Long Division

4) $x^5 + 4x^4 + 6x^3 + 18x^2 - 27x - 162 \div (x^2 + 9)$

$$x^3 + 4x^2 - 3x - 18$$

$$(x^2 + 9)\ \overline{)\ x^5 + 4x^4 + 6x^3 + 18x^2 - 27x - 162}$$

$$-\ x^5 \quad + 9x^3$$

$$4x^4 - 3x^3 + 18x^2$$

$$-\ 4x^4 \quad + 36x^2$$

$$-3x^3 - 18x^2 - 27x$$

$$-\ -3x^3 \quad -27x$$

$$-18x^2 \quad - 162$$

$$-\ -18x^2 \quad - 162$$

$$0$$

$$\boxed{x^3 + 4x^2 - 3x - 18}$$

5) $x^2 - 4x - 2 \div [x - (3 + 4i)]$

$$x - 1$$

$$[x - (3 + 4i)]\ \overline{)\ x^2 - 4x - 2}$$

$$-\ x^2 - 3x \quad - 4xi$$

$$-x - 2 + 4xi$$

$$-\ -x + 3 \quad + 4i$$

$$-5 + 4xi - 4i$$

$$\boxed{x + 4i - 1 + \frac{-21 + 8i}{x - (3 + 4i)}}$$

$$4i \cdot \frac{x - (3 + 4i)}{x - (3 + 4i)}$$

$$\frac{4ix - 12i - 16i^2}{x - (3 + 4i)} + \frac{-21 + 8i}{x - (3 + 4i)} = \frac{-5 + 4xi - 4i}{x - (3 + 4i)}$$

(using the remainder theorem)

$$f(3 + 4i) = (3 + 4i)^2 - 4(3 + 4i) - 2$$

$$9 + 24i + 16i^2 - 12 - 16i - 2$$

$$8i - 21$$ ✓

(using synthetic division)

$3 + 4i$	1	-4	-2
		$3 + 4i$	$-19 + 8i$
	1	$-1 + 4i$	$-21 + 8i$

$(3 + 4i)(-1 + 4i) =$

$-3 + 12i - 4i + 16i^2$

$-19 + 8i$

6) $x^6 - 4x^3 + 6x^2 + 10 \div (2x^2 + 5)$

$$\frac{1}{2}x^4 \quad - \frac{5}{4}x^2 - 2x + \frac{49}{8}$$

$$(2x^2 + 5)\ \overline{)\ x^6 + 0x^5 + 0x^4 - 4x^3 + 6x^2 + 0x + 10}$$

$$-\ x^6 \quad + \frac{5}{2}x^4$$

$$-\frac{5}{2}x^4 - 4x^3 + 6x^2$$

$$-\ -\frac{5}{2}x^4 \quad - \frac{25}{4}x^2$$

$$-4x^3 + \frac{49}{4}x^2$$

$$-\ -4x^3 \quad - 10x$$

$$\frac{49}{4}x^2 + 10x + 10$$

$$-\ \frac{49}{4}x^2 \quad + \frac{245}{8}$$

$$10x - \frac{165}{8}$$

$$\boxed{\frac{1}{2}x^4 - \frac{5}{4}x^2 - 2x + \frac{49}{8} + \frac{10x - \frac{165}{8}}{(2x^2 + 5)}}$$

SOLUTIONS

1) Find the factors of $x^4 + x^3 - x^2 + x - 2$

Possible rational roots are 1, -1, 2, -2...

$f(1) = 1 + 1 - 1 + 1 - 2 = 0$ so, 1 is a root...

1	1	1	-1	1	-2
		1	2	1	2
	1	2	1	2	0

$(x - 1)$

$x^3 + 2x^2 + x + 2$

(recognizing a pattern, we can use grouping to factor the remaining part.)

$x^3 + 2x^2 \quad + x + 2$

$x^2(x + 2) \quad + 1(x + 2)$

$(x^2 + 1)(x + 2)$

$(x - 1)(x + 2)(x^2 + 1)$

2) Find a 4th degree polynomial with zeros $3i$ and -1 , -1 has a multiplicity of 2.

Since $3i$ is a zero, $-3i$ must be a zero... (conjugate root theorem)

Since -1 has a multiplicity of 2, there are two zeros that are -1...

$3i \quad -3i \quad -1 \quad -1$

$(x - 3i)(x + 3i)(x + 1)(x + 1)$

$(x^2 + 9)(x + 1)^2$

3) A) Write a quadratic equation in standard form whose solutions are $(2 + 5i)$ and $(2 - 5i)$

$(x - (2 + 5i)(x - (2 - 5i)$

$(x - 2 - 5i)(x - 2 + 5i)$

x	$(x - 2 + 5i)$	$x^2 - 2x + 5ix$
-2	$(x - 2 + 5i)$	$-2x \quad + 4 - 10i$
$-5i$	$(x - 2 + 5i)$	$-5ix \quad + 10i - 25i^2$
		$x^2 - 4x \quad + 4 \quad + 25$

$x^2 - 4x + 29$

B) Now, write a cubic equation whose solutions (zeros) are -1, $2 + 5i$, and $2 - 5i$

$-1 \quad 2 + 5i \quad 2 - 5i$

$(x + 1)(x - (2 + 5i)(x - (2 - 5i)$

$(x + 1)(x^2 - 4x + 29)$ or $x^3 - 3x^2 + 25x + 29$

C) And, finally, write a quartic equation in factored form whose solutions (zeros) are -1, $2 + 5i$, and $2 - 5i$

Since complex numbers must come in pairs,
the fourth zero must be -1...

$(x + 1)(x + 1)(x - (2 + 5i)(x - (2 - 5i)$

$(x + 1)^2(x - 2 - 5i)(x - 2 + 5i)$

4) Using synthetic division, determine the following:

A) $x^3 + 7x - 1 \div (x - 2)$

2	1	0	7	-1
		2	4	22
	1	2	11	21

$x^2 + 2x + 11 + \frac{21}{(x - 2)}$

B) $3x^3 - 2x^2 + 5x + 1 \div (3x + 1)$

first, rewrite: $(3x^3 - 2x^2 + 5x + 1) \div 3(x + \frac{1}{3})$

If we separate the 3,

$(3x^3 - 2x^2 + 5x + 1) \div (x + \frac{1}{3})$

$\frac{-1}{3}$	3	-2	5	1
		-1	1	-2
	3	-3	6	-1

$3x^2 - 3x + 6 - \frac{1}{(x + \frac{1}{3})}$

Then, divide by the 3...

$x^2 - x + 2 - \frac{1}{(3x + 1)}$

5) Why can't +1 be a possible root of

$$f(x) = 3x^4 + x^3 + 9x^2 + 6x + 15 \ ?$$

(in other words, why is (x − 1) NOT a factor?)

According to factor/remainder theorem, $f(1)$ must equal 0 in order for 1 to be a root. Since all the polynomial terms are > 0, there is no way synthetic substitution will lead to 0! (the result must be greater than 0)

SOLUTIONS

6) $g(x) = 2x^3 - 10x^2 + 200x - 1000$

Consider the possible rational roots.
Which ones should you omit?
Which should you test first?
Explain.

Since the degree of the trinomial is 3, there are up to 3 possible rational roots.
The possible roots include all factors of 1000. (positive and negative).
divided by 1 and 2

Since $g(c)$ must equal zero if c is a root, we can eliminate 1 and -1
(result is much less than zero)

$-10x^2 + 200x - 1000$ will end in zero for all integers

we can skip 2, -2, 4, -4, 8, -8
(because the results will not end in zero)

So, start with 5 and -5

7) Find the zeroes: $f(x) = x^6 + 6x^5 + 15x^4 + 36x^3 + 54x^2$

(factor the GCF) $x^2(x^4 + 6x^3 + 15x^2 + 36x + 54)$

(rational root theorem) possible roots are 1, 2, 3, 6, 9, 18, 27, 54
-1, -2, -3, -6, -9, -18, -27, -54

since all the coefficients are positive, this eliminates all positive roots...
so, possible roots are -1, -2, -3, -6, -9, -18, -27, -54

test -1: 1 - 6 + 15 - 36 + 54 = 28... remainder is 28 (not a root)

test -2: 16 - 48 + 60 - 72 + 54 = 10... remainder is 10 (not a root)

test -3: 81 - 162 + 135 - 108 + 54 = 0... remainder is 0 (factor!)

-3	1	6	15	36	54
		-3	-9	-18	-54
	1	3	6	18	0

$x^3 + 3x^2 + 6x + 18$

factors: $x^2(x + 3)^2\ (x^2 + 6)$

zeros: 0 (multiplicity of 2) ; -3 (multiplicity of 2)
and, $\sqrt{6}$ $-\sqrt{6}$

test -3: -27 + 27 - 18 + 18 = 0
remainder is 0 (-3 is a zero again)

-3	1	3	6	18
		-3	0	-18
	1	0	6	0

$x^2 + 6$

remaining zeros are $-\sqrt{6}$ and $\sqrt{6}$

8) Let $g(x) = 2x^3 - 5x^2 - 4x + 3$

Find the *possible* rational zeros of g

'p's: 1, 3
'q's: 1, 2

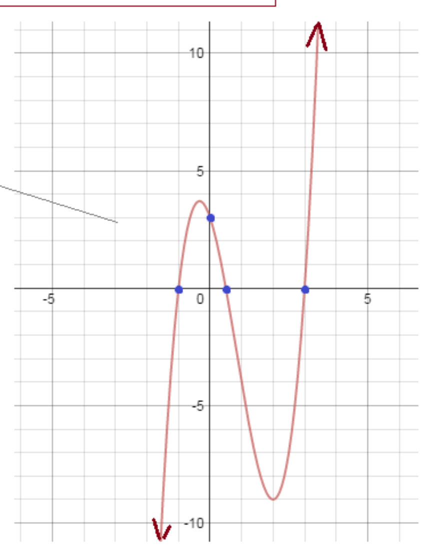
possible rational roots: 1, -1, 3, -3, 1/2, -1/2, 3/2, -3/2

Determine the complete factorization

What are the x-intercepts? The y-intercept?

degree is 3
end behavior is "up right"
and "down left"
x-intercepts and y-intercept
are labeled

Sketch the graph



$g(-1) = 2(-1)^3 - 5(-1)^2 - 4(-1) + 3 = 0$ -1 is a zero; (x + 1) is a factor

-1	2	-5	-4	3
		-2	7	-3
	2	-7	3	0

$2x^2 - 7x + 3 = (2x - 1)(x - 3)$

factored form: $(2x - 1)(x - 3)(x + 1)$

x-intercepts: (-1, 0) (1/2, 0) (3, 0)
y-intercept: (0, 3)

Polynomials Concepts: Graphing

SOLUTIONS

Sketch the following polynomial:

$$y = \frac{1}{10}(x+3)(x-2)(x-5)^2$$

Label the intercepts...

The above polynomial is in *factored form* or *intercept form*, so the x-intercepts (zeros) are shown:

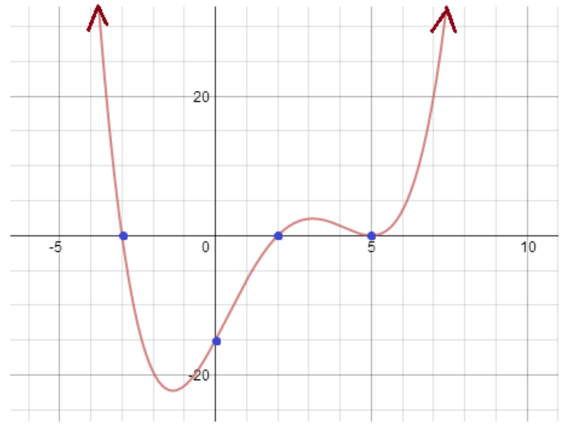
(-3, 0)   (2, 0)   (5, 0)

Note: (x - 5) has a multiplicity of 2, so there will be a "bounce" at (5, 0)

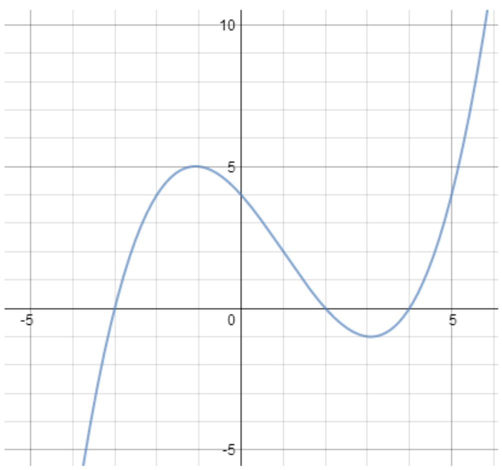
The y-intercept occurs when the function is at x = 0.   (0, ?)

$$y = \frac{1}{10}(0+3)(0-2)(0-5)^2 = -15$$

y-intercept: (0, -15)



Write the equation of the polynomial in the following graph:



Step 1: Identify the x-intercepts

(-3, 0)   (2, 0)   (4, 0), so the 'zeros' are -3, 2, and 4..

$$y = a(x+3)(x-2)(x-4)$$

Step 2: Note the end behavior and any 'bounces'...

There are no 'bounces' or multiplicity of the zeros....
And, the end behavior indicates a cubic...

Step 3: Find the "a" value by substituting another point...

Using (0, 4) $4 = a(0+3)(x-2)(x-4)$

$4 = 24a$

$a = 1/6$

$$y = \frac{1}{6}(x+3)(x-2)(x-4)$$

Write the equation of the polynomial in the following graph:

Step 1: Identify the x-intercepts

(-2, 0)   (1, 0) and (4, 0), so the zeros are -2, 1, and 4

$$y = a(x+2)(x-1)(x-4)$$

Step 2: Note the end behavior and any 'bounces'...

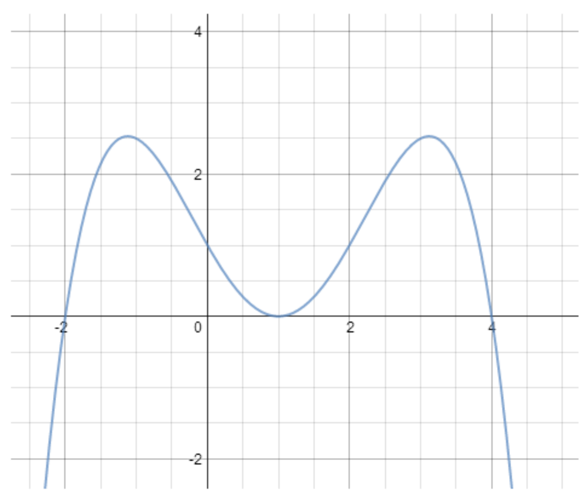
There is a "bounce" at (1, 0) ---> multiplicity of 2
The end behavior indicates a polygon of degree 4

$$y = a(x+2)(x-1)^2(x-4)$$

Step 3: Find the "a" value by substituting another point...

using (0, 1) $1 = a(0+2)(0-1)^2(0-4)$ $1 = -8a$ $a = -1/8$

$$y = -\frac{1}{8}(x-1)^2(x-4)(x+2)$$



The following are results of synthetic division from the function Q(x) and selected linear binomials....

Polynomials: Concepts and Synthetic Division

SOLUTIONS

−5	1	-3	1	-15	-20
		-5	40	-205	1100
	1	-8	41	-220	1080

1	1	-3	1	-15	-20
		1	-2	-1	-16
	1	-2	-1	-16	-36

−4	1	-3	1	-15	-20
		-4	28	-116	524
	1	-7	29	-131	504

2	1	-3	1	-15	-20
		2	-2	-2	-34
	1	-1	-1	-17	-54

−3	1	-3	1	-15	-20
		-3	18	-57	216
	1	-6	19	-72	196

3	1	-3	1	-15	-20
		3	0	3	-36
	1	0	1	-12	-56

−2	1	-3	1	-15	-20
		-2	10	-22	74
	1	-5	11	-37	54

4	1	-3	1	-15	-20
		4	4	20	20
	1	1	5	5	0

−1	1	-3	1	-15	-20
		-1	4	-5	20
	1	-4	5	-20	0

5	1	-3	1	-15	-20
		5	10	55	200
	1	2	11	40	180

1) What is Q(3)? (Utilize the remainder theorem!)

$Q(3) = -56$ It's the remainder of $Q(x) \div (x-3)$

2) Find $\frac{Q(x)}{x+2}$ (Recognizing Synthetic Division)

$\frac{Q(x)}{x+2}$ → -2 | 1 -5 11 -37 | 54 → $x^3 - 5x^2 + 11 - 37 + \frac{54}{x+2}$

3) Find the equation of the polynomial Q(x) in *standard form.*

Pick any from above, convert to polynomials, and multiply!

EX: -2 | 1 -5 11 -37 | 54

$(x+2)(x^3 - 5x^2 + 11x - 37 + \frac{54}{(x+2)})$

$x^4 + 2x^3 - 5x^3 - 10x^2 + 11x^2 + 22x - 37x - 74 + 54$

$Q(x) = x^4 - 3x^3 + x^2 - 15x - 20$

4) Write the equation of the polynomial as a product of linear factors (or factored form)

Pick out one of the roots (i.e. remainder is 0):

either -1 or 4

-1	1	-3	1	-15	-20
		-1	4	-5	20
	1	-4	5	-20	0

Then, divide by the other root:

4	1	-4	5	-20
		4	0	20
	1	0	5	0

$x^2 + 5$

AND

the roots/zeros are -1 and 4

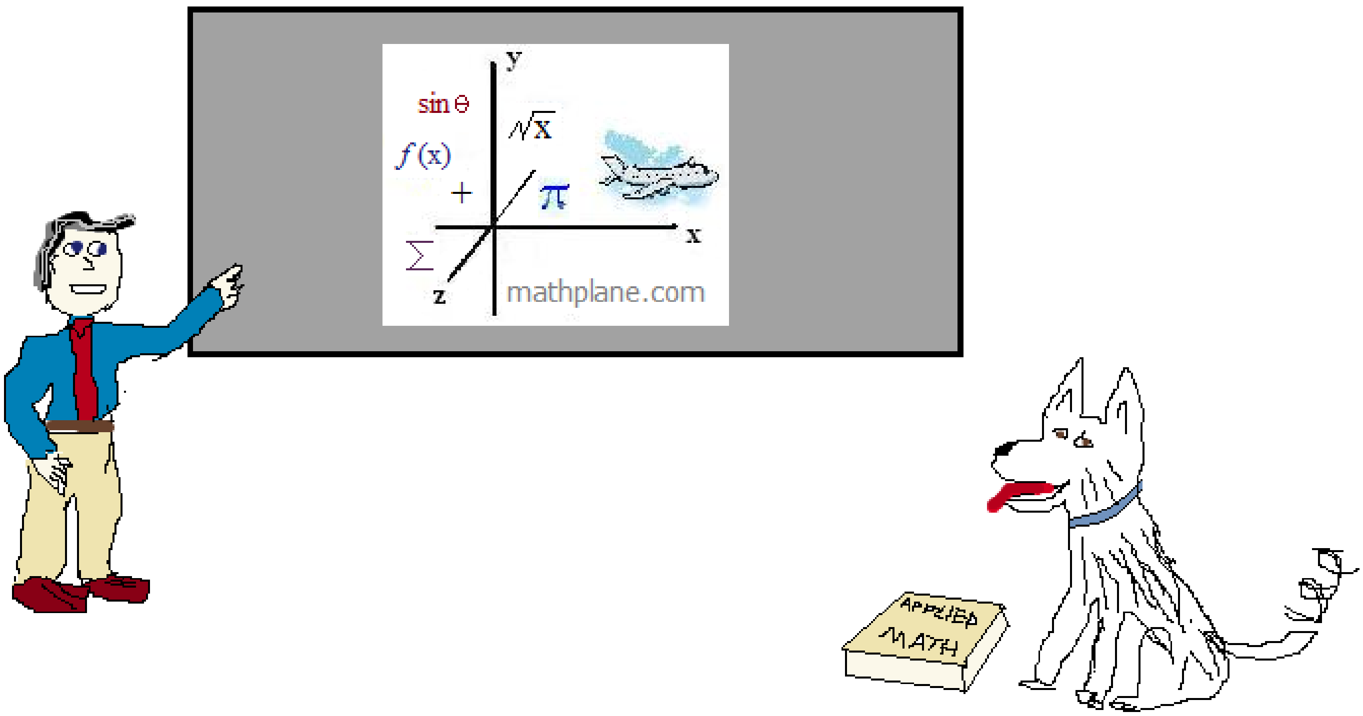
$(x+1)(x-4)(x^2+5)$

or, $(x+1)(x-4)(x+\sqrt{5}\,i)(x-\sqrt{5}\,i)$

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Enjoy.



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