Geometric Mean and Proportional Right Triangles

Notes, Examples, and Practice Exercises (with Solutions)

Topics include geometric mean, similar triangles, Pythagorean Theorem, and more.
Cross Product & Similar Right Triangles

Using Cross Products to compare fractions

If \( \frac{A}{B} = \frac{C}{D} \) then \( AD = BC \)

\[ \frac{3}{4} = \frac{12}{16} \quad \text{---}> \quad 3 \times 16 = 4 \times 12 = 48 \]

\[ \frac{A}{B} = \frac{C}{D} \quad \text{multiply both sides by } B \quad A = \frac{BC}{D} \quad \text{multiply both sides by } D \quad AD = BC \]

If \( \frac{A}{B} = \frac{C}{D} \) then \( \frac{A}{C} = \frac{B}{D} \)

\[ \frac{5}{9} = \frac{25}{45} \quad \text{---}> \quad \frac{5}{25} = \frac{9}{45} \]

\[ \frac{A}{B} = \frac{C}{D} \quad \text{multiply both sides by } B \quad A = \frac{BC}{D} \quad \text{divide both sides by } C \quad \frac{A}{C} = \frac{B}{D} \]

Application: similar right triangles

For these similar triangles, the above ratios apply!
Notes on Means-Extremes, Proportions, & Right Triangles

Draw an altitude to hypoteneuse. 
Three similar right triangles are formed.

△ABC ∼ △BDC ∼ △ADB
3 similar triangles: each pair can be proven using (AA) Angle-Angle -- Triangle Similarity Theorems

Since the right triangles are similar, the ratios of their sides are the same.

\[
\frac{\text{left leg}}{\text{hypoteneuse}} = \frac{AB}{AC} = \frac{BD}{BC} = \frac{AD}{DB}
\]

\[
\frac{\text{left leg (big)}}{\text{left leg (med)}} = \frac{AB}{BD} = \frac{AC}{BC} = \frac{\text{hypo (big)}}{\text{hypo (med)}}
\]

\[
\frac{AB}{AD} = \frac{AC}{AB} \rightarrow AB^2 = AC \cdot AD
\]

(note: using triangle similarity ratios, one can derive the pythagorean theorem)
Special Right Triangles

Review Notes:

Pythagorean Theorem: \( a^2 + b^2 = c^2 \)

![Diagram of a right triangle with sides labeled a, b, and c, opposite side, hypotenuse, and adjacent sides labeled]

Trigonometry Relations:

\[
\begin{align*}
\sin \theta &= \frac{b}{c} & \csc \theta &= \frac{c}{b} \\
\cos \theta &= \frac{a}{c} & \sec \theta &= \frac{c}{a} \\
\tan \theta &= \frac{b}{a} & \cot \theta &= \frac{a}{b}
\end{align*}
\]

Utilizing the Pythagorean Theorem or Trig Identities can find angle and side measurements. However, "Special Right Triangles" have features that made calculations easy!!

Special Right Triangles:

"Sides"

3 - 4 - 5 Right Triangle

- 4
- 3
- 5

Others include: 5 - 12 - 13
- 7 - 24 - 25
- 8 - 15 - 17

"Angles"

30 - 60 - 90 Right Triangle

- 30°
- 60°
- 90°

1
\( \sqrt{3} \)

Note:
-- Pythagorean theorem confirms
\( 3^2 + 4^2 = 5^2 \)
-- Any multiple of 3-4-5 will work!
Examples: 30-40-50 or 15-20-25

Note:
-- Pythagorean theorem and trig relations confirm
\( \text{ex: } \sin 30^\circ = 1/2 = .5 \)
-- any ratio of 1 - \( \sqrt{3} \) - 2 will work.
\( \rightarrow x - \sqrt{3} x - 2x \)

Note:
-- Pythagorean theorem and trig relations confirm
-- Congruent sides imply congruent (opposite) angles
-- any ratio of 1 - 1 - \( \sqrt{2} \) will work.
\( \rightarrow x - x - \sqrt{2} x \)
**Right Triangles: Altitude, Geometric Mean, and Pythagorean Theorem**

*Example:* Find $x$:

![Diagram of a right triangle with sides 3, 9, and x.]

Step 1: Find the length of the altitude...

\[ \frac{3}{h} = \frac{h}{9} \quad h = \sqrt{27} \]

**Geometric mean of divided hypotenuse is the length of the altitude**

\[ \sqrt{27} \] is the geometric mean of $3$ and $9$.

Step 2: Find $x$

\[ \sqrt{27}^2 + 9^2 = x^2 \]
\[ 27 + 81 = x^2 \]
\[ x = \sqrt{108} \]

**Pythagorean Theorem:**

\[ a^2 + b^2 = c^2 \] where $a$ and $b$ are legs and $c$ is the hypotenuse.

Step 3: Check solution (with other sides)

\[ 3^2 + \sqrt{27}^2 = c^2 \]
\[ c = 6 \]

Then,

\[ 6^2 + \sqrt{108}^2 = 12^2 \]

\[ 36 + 108 = 144 \quad \checkmark \]

(all 3 right triangles satisfy the Pythagorean Theorem)
A surfer wants to walk directly to the beach from his car. (see diagram)

a) What is the shortest distance to the beach?

b) How far is the beach spot from the snack bar?

*** The walk directly to the beach will form a right angle (i.e. creating altitude to hypotenuse)

*** The distance from Restroom to Snack Bar is 100 yds. (Pythagorean Theorem)

a) Recognizing "altitude to hypotenuse" cuts right triangle into 3 similar right triangles....

\[
\frac{\text{hypotenuse}}{\text{small leg}} = \frac{80}{d} = \frac{100}{60}
\]

\[d = 48\]

b) Then, to find distance from beach spot to snack bar (x) we know that d is the geometric mean between x and 100 - x...

\[\frac{100 - x}{d} = \frac{d}{x}\]
\[\frac{100 - x}{48} = \frac{48}{x}\]

Distance from beach spot to snack bar is 64, because \(64^2 + 48^2 = 80^2\)

\[2304 = 100x - x^2\]
\[x^2 - 100x + 2304 = 0\]
\[x = 36 \text{ or } 64\]
"Do you know which part they are hiring?"

"I'm not sure. I hope I'm the one they're looking for!"

"I see you had a part in 'Toy Story 2'... and a brief stint in Mrs. Bern's 2nd grade math class..."

"...yes, and, I recently posed in a 2 for 1 sales commercial."

"Psst... Hey, Count... Are there any more cookies?"

Auditioning for the role of a lifetime!
1) \[ \frac{DB}{AC} \]
\[ AD \perp CD \]
\[ BC = 5 \]
\[ AD = 6 \]

Find the length \( DB \)
and \( AB \).

2) Write a similarity statement for the 3 triangles:

3) Given Trapezoid \( TRAP \), with bases \( TR \) and \( PA \):
Find \( TR \) and \( RA \).
Parts of Proportional Right Triangles

Find x:

A)

\[
x + 3
\]

9 \hspace{1cm} x

B)

\[
2x + 1
\]

8 \hspace{1cm} x

C)

\[
x - 2
\]

6 \hspace{1cm} 10

D)

\[
x - 10 \quad \quad \quad \quad \quad 11
\]

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Parts of Proportional Right Triangles

Geometric mean of divided hypotenuse is the length of the altitude

Solve:

1)

\[
\begin{array}{c}
30 \\
6 \\
x \\
y \\
z
\end{array}
\]

2)

\[
\begin{array}{c}
x + 4 \\
x \\
12
\end{array}
\]

3)

\[
\begin{array}{c}
x \\
y \\
z \\
16 \\
9
\end{array}
\]
SOLUTIONS

1) Find the length $\overline{DB}$ and $\overline{AB}$

\[
\overline{DB} = 2 \sqrt{5} \\
\overline{AB} = 4
\]

2) Write a similarity statement for the 3 triangles:

\[\triangle ABD \sim \triangle DBC \sim \triangle ADC\]

The similar triangles must correspond!

ex: $\triangle ABD$ is not similar to $\triangle CBD$

3) Given Trapezoid $\text{TRAP}$, with bases $\overline{TR}$ and $\overline{PA}$...

Find $\overline{TR}$ and $\overline{RA}$

First, draw altitudes to create right triangles.

then, using geometry properties, label the other parts.

\[
\overline{TR} = 7 + 4\sqrt{3} \\
\overline{RA} = 4\sqrt{6}
\]

30-60-90 rt triangle $1 : \sqrt{3} : 2$
Parts of Proportional Right Triangles

Find x:

A)

\[ Y = \sqrt{9x} \]  
(altitude is geometric mean of split hypotenuse)

\[ Y = \sqrt{(x + 3)^2 - x^2} \]  
(Pythagorean Theorem)

\[ \sqrt{9x} = \sqrt{(x + 3)^2 - x^2} \]  
(substitution)

\[ 9x = x^2 + 6x + 9 - x^2 \]

\[ 8x = 9 \]

\[ x = 3 \]

B)

\[ R^2 + 8^2 = (2x + 1)^2 \]  
(Pythagorean Theorem)

\[ R^2 = 8x \]  
(Geometric mean of altitude)

\[ \frac{8}{R} = \frac{R}{x} \]

C)

1) Pythagorean Thm: \(6^2 + 8^2 = 10^2\)

2) Alt. to Hypotenuse: 6 is geometric mean of y and 8

\[ 6^2 = 8y \]

then, \( y = \frac{36}{8} = 9/2 \)

D)

Altitude to hypotenuse:

\[ 12^2 - 11(x - 10) \]

\[ 144 - 11x - 110 \]

\[ 11x = -254 \]

\[ x = 23.1 \]

SOLUTIONS

\[ Y = \sqrt{9x} \]  
(altitude is geometric mean of split hypotenuse)

\[ Y = \sqrt{(x + 3)^2 - x^2} \]  
(Pythagorean Theorem)

\[ \sqrt{9x} = \sqrt{(x + 3)^2 - x^2} \]  
(substitution)

\[ 9x = x^2 + 6x + 9 - x^2 \]

\[ 8x = 9 \]

\[ x = 3 \]

Set equations equal to each other:

\[ (2x + 1)^2 - 8^2 = 8x \]

\[ 4x^2 + 4x + 1 - 64 = 8x \]

\[ 4x^2 - 4x - 63 = 0 \]

\[ (2x - 9)(2x + 7) = 0 \]

\[ x = 9/2 \] or \(-7/2\)

Since x cannot be negative, the solution is

\[ x = \frac{9}{2} \] or \(4.5\)

To check: See if all the right triangle measures are OK

3) Pythagorean Thm: \(6^2 + (9/2)^2 = (x - 2)^2\)

\[ 36 + 81/4 = x^2 - 4x + 4 \]

\[ x^2 - 4x - 52.25 = 0 \]

\[ x = 9.5 \] or \(-5.5\) (quadratic formula)

Since a side cannot be negative, \(x = 9.5\)

To check: observe all the right triangles:

6-8-10  4.5-6-7.5  7.5-10-12.5

2 \(x \) (3-4-5)  1.5 \(x\) (3-4-5)  2.5 \(x\) (3-4-5)

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**Parts of Proportional Right Triangles**

**Solutions**

1) \(x^2 = (24)(6)\)
   \[x = 12\]
   \[x^2 + 6^2 = y^2\]
   \[144 + 36 = y^2\]
   \[y = \sqrt{180} = 6\sqrt{5}\]
   \[y^2 + z^2 = 30^2\]
   \[180 + z^2 = 900\]
   \[z = \sqrt{720} = 12\sqrt{5}\]

2) \(Y^2 = (x + 4)^2 - x^2\)
   \[Y^2 = (x)(12 - x)\]
   \[(x + 4)^2 - x^2 = (x)(12 - x)\]
   \[8x + 16 = 12x - x^2\]
   \[x^2 - 4x + 16 = 0\]
   \[\text{NO SOLUTION!!}\]

3) \(\frac{y}{9} = \frac{16}{y}\)
   \[y = 12\]
   \[z = 15\]
   \[3 \times (3 - 4 - 5) = 9 - 12 - 15 \text{ right triangle}\]
   \[x^2 + z^2 = 25^2\]
   \[x^2 + 225 = 625\]
   \[x = 20\]

**Geometric mean of divided hypotenuse is the length of the altitude**
Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Enjoy

Also, at Facebook, Google+, TeachersPayTeachers, and Pinterest

One more example →
Find the length of $\overline{DE}$

**Step 1: Utilize the "Geometric Mean of divided Hypotenuse"

\[
\frac{AD}{DC} = \frac{DC}{DB}
\]

\[
DC^2 = AD \cdot DB
\]

\[
DC = \sqrt{24}
\]

**Step 2: Utilize the Pythagorean Theorem

\[
DB^2 + DC^2 = CB^2
\]

\[
64 + 24 = CB^2
\]

\[
CB = \sqrt{88}
\]

\[
CB^2 + AC^2 = AB^2
\]

\[
88 + AC^2 = 121
\]

\[
AC = \sqrt{33}
\]

**Step 3: Use the "Angle Bisector Theorem"

Since $AE$ is an angle bisector in triangle $CAD$,

\[
\frac{AD}{AC} = \frac{DE}{CE}
\]

\[
\frac{3}{\sqrt{33}} = \frac{x}{\sqrt{24} - x}
\]

\[
3\sqrt{24} - 3x = \sqrt{33}x
\]

\[
3\sqrt{24} = \sqrt{33}x + 3x
\]

\[
14.697 = 8.745x
\]

\[
x = 1.68
\]