# Geometry: Proofs and Postulates

Definitions, Notes, & Examples

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<td>1. Given</td>
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<td>2. $\overline{AM} \cong \overline{DM}$ ; $\overline{CM} \cong \overline{BM}$</td>
<td>2. Definition of bisector</td>
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<td>3. $\angle AMC \cong \angle BMD$</td>
<td>3. Vertical angles are congruent</td>
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<td>4. $\triangle AMC \cong \triangle DMB$</td>
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<td>5. $\overline{AC} \cong \overline{BD}$</td>
<td>5. CPCTC (Corresponding Parts of Congruent Triangles are Congruent)</td>
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Topics include triangle characteristics, quadrilaterals, circles, midpoints, SAS, and more.
Proofs and Postulates: Triangles and Angles

Postulate: A statement accepted as true without proof.

I. A **Straight Angle** is **180°**

![Diagram of a straight angle](image)

A circle has **360°**

![Diagram of a circle with 360° labeled](image)

It follows that the semi-circle is **180 degrees**.

II. **Supplementary Angles** add up to **180°**

\[ \text{m} \angle A + \text{m} \angle B = 180° \]

![Diagram showing supplementary angles](image)

**Example:**

![Diagram showing supplementary angles](image)

:\( \angle xyr \) and \( \angle xyz \) are supplementary angles.

And, although they are not adjacent, \( \angle S \) and \( \angle xyr \) are supplementary as well.

**Theorem:** A statement or assertion that can be proven using rules of logic.

III. **Vertical Angles** are congruent

![Diagram of vertical angles](image)

\[ \angle R \cong \angle S \quad \angle X \cong \angle Y \]

**Examples:**

![Diagram showing examples of vertical angles](image)

\( \text{Informal proof: } \angle A = \angle C \)

\( A + B = 180 \text{ degrees} \) (supplementary angles)

\( B + C = 180 \text{ degrees} \) (supplementary angles)

\( A = C \) (substitution)

Using postulates and math properties, we construct a sequence of logical steps to prove a theorem.
Parallel Line Postulate: If 2 parallel lines are cut by a transversal, then their corresponding angles are congruent.

\[
\angle 1 \cong \angle 2
\]

IV. If parallel lines are cut by a transversal, the alternate interior angles are congruent (Theorem)

\[
l \parallel m
\]

\[
\angle A \cong \angle B
\]

Proof of parallel lines/alt. interior angles:

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<td>1. ( l \parallel m )</td>
<td>1. given</td>
</tr>
<tr>
<td>2. ( t ) is transversal</td>
<td>2. given (def. of transversal)</td>
</tr>
<tr>
<td>3. ( \angle D \cong \angle E )</td>
<td>3. if parallel lines cut by transversal, then corresponding angles are congruent</td>
</tr>
<tr>
<td>4. ( \angle C \cong \angle D )</td>
<td>4. vertical angles congruent</td>
</tr>
<tr>
<td>5. ( \angle C \cong \angle E )</td>
<td>5. substitution</td>
</tr>
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Examples:

If \( \angle 2 = 70^\circ \) and \( r \) is parallel to \( s \),

4 = 110° (2 and 4 are supplementary)
3 = 70° (3 and 2 are corresponding)
5 = 70° (3 and 5 are vertical angles)
6 = 70° (3 and 6 are alt. interior angles)
1 = ? (\( p \) is not parallel to \( r \) or \( s \))
V. The sum of the interior angles of a triangle is $180^\circ$ (Theorem)

$m \angle A + m \angle B + m \angle C = 180^\circ$

**Examples:**

$x + 43 + 85 = 180$ degrees
$x = 52$ degrees

$S + 40 + 80 = 180$
$S = 60$ degrees

$T + S = 180$ degrees
$T + 60 = 180$
So, $T = 120$ degrees

**Illustrates the triangle (remote) exterior angle theorem:** the measure of an exterior angle equals the sum of the 2 non-adjacent interior angles.

**Informal Proof:** $1 + 2 + 3 = 180^\circ$
Add parallel line to one of the sides

$A + 1 + B = 180$ degrees (straight angle and addition postulate)
$A = 2$ and $B = 3$ (parallel lines cut by transversal, then alt. interior angles are congruent)
$2 + 1 + 3 = 180$ degrees (substitution)
Tools to consider in Geometry proofs:

1) Using CPCTC (Corresponding Parts of Congruent Triangles are Congruent) after showing triangles within the shapes are congruent.
   Try
   a) reflexive property
   b) vertical angles are congruent
   c) alternate interior angles (formed by parallel lines cut by a transversal) are congruent

Then, verify congruent triangles by SAS, SSS, ASA, AAS, HL.

2) Common properties and theorems
   a) Triangles are 180°; Quadrilaterals are 360°
   b) Opposite sides of congruent angles are congruent (isosceles triangle)
   c) Perpendicular bisector Theorem
      (All points on perpendicular bisector are equidistant to endpoints)

   [Diagram]

   DM is perpendicular bisector of BC
   (B and C are the endpoints)

   \[
   \frac{BD}{CD} = \frac{BA}{CA}
   \]

3) Other geometry basics:
   a) All radii of a circle are congruent
   b) Supplementary angles (180°); Complementary angles (90°)
   c) Midpoints and medians divide segments into congruent parts
   d) Altitudes form right angles

   [Diagram]

   \[
   \overline{AM} \text{ is median of } \triangle ABC
   \]

   \[
   M \text{ is midpoint of } BC
   \]

   \[
   \overline{DR} \text{ is altitude of } \triangle DEF
   \]

   \[
   \angle DRE \text{ and } \angle DRF \text{ are right angles}
   \]
... Stranded somewhere in the (Bermuda) Triangle...

... the Math Guy -- losing his mind -- mistakenly builds a geometry message

SAS is not a distress signal

When you're in the Bermuda Triangle, SOS is more useful than SAS!!
Proving a Median of a Triangle: Example

Given: \( \overline{CM} \) bisects \( \angle BCD \)
\[ \overline{DC} \cong \overline{BC} \]
Prove: \( \overline{AM} \) is a median of \( \triangle BDA \)

Step 1: "Label the picture."
\( \overline{CM} \) bisects \( \angle BCD \)
\[ \overline{DC} \cong \overline{BC} \]

What are we trying to prove? \( \overline{DM} \cong \overline{MB} \)
(definition of a median)

Step 2: Determine a Strategy
To prove a median, I need to show a segment bisects
the opposite side. (def. of a median).

Notice triangles \( \triangle CMD \) & \( \triangle CMB \). They include \( \overline{DM} \) & \( \overline{MB} \).
If I can show \( \triangle CMD = \triangle CMB \), then I can use CPCTC to
prove that \( \overline{DM} = \overline{MB} \).

Step 3: Write the Proof (describing your approach
and strategy)

<table>
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<tr>
<td>1) ( \overline{DC} = \overline{BC} ) &lt;br&gt; ( \overline{CM} ) bisects ( \angle BCD )</td>
<td>1) Given</td>
</tr>
<tr>
<td>2) ( \angle DCM = \angle BCM )</td>
<td>2) Definition of Angle Bisector</td>
</tr>
<tr>
<td>3) ( \overline{MC} = \overline{MC} )</td>
<td>3) Reflexive Property</td>
</tr>
<tr>
<td>4) ( \triangle DCM = \triangle BCM )</td>
<td>4) Side-Angle-Side (SAS) postulate</td>
</tr>
<tr>
<td>5) ( \overline{DM} = \overline{BM} )</td>
<td>5) Corresponding Parts of Congruent Triangles are Congruent (CPCTC)</td>
</tr>
<tr>
<td>6) ( \overline{AM} ) is median of ( \triangle ABD )</td>
<td>6) Definition of a Median (A line segment joining vertex of triangle to midpoint of opposite side)</td>
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</table>
"Prove the base angles of an isosceles triangle are congruent."

Given: Isosceles Triangle
Prove: Base Angles are congruent

Step 1: Draw pictures and label.
   What is given? A triangle with 2 sides of equal length. (Definition of an Isosceles Triangle)

   What are we trying to prove? ∠B ≅ ∠C

Step 2: Determine a strategy.
   Try dividing into triangles.
   Use properties of congruent triangles.
   Then, CPCTC.

Step 3: Write the proof (describing your strategy!)

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<tr>
<td>1. $AB \cong AC$</td>
<td>1. Given/Definition of Isosceles Triangle</td>
</tr>
<tr>
<td>2. Draw a Median</td>
<td>2. An angle has one median</td>
</tr>
<tr>
<td>3. $BM \cong CM$</td>
<td>3. Definition of a median (A line segment joining a vertex and the midpoint of the opposite side)</td>
</tr>
<tr>
<td>4. $AM \cong AM$</td>
<td>4. Reflexive Axiom</td>
</tr>
<tr>
<td>5. $\triangle AMB = \triangle AMC$</td>
<td>5. SSS congruence postulate (If 3 sides of one triangle are congruent to corresponding sides of another triangle, then the triangles are congruent)</td>
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<tr>
<td>6. $\angle B \cong \angle C$</td>
<td>6. CPCTC (Corresponding Parts of Congruent Triangles are Congruent)</td>
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Alternate Proof: (Using angle bisector and Side-Angle-Side)

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<tr>
<td>1. $\triangle ABC$; $AB \cong AC$</td>
<td>1. Given; Def. of Isosceles triangle</td>
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<tr>
<td>2. $AM$ is an angle bisector</td>
<td>2. An angle has one bisector</td>
</tr>
<tr>
<td>3. $\angle BAM \cong \angle CAM$</td>
<td>3. Def. of Angle Bisector</td>
</tr>
<tr>
<td>4. $AM \cong AM$</td>
<td>4. Reflexive Property</td>
</tr>
<tr>
<td>5. $\triangle BAM = \triangle CAM$</td>
<td>5. SAS postulate</td>
</tr>
<tr>
<td>6. $\angle B = \angle C$</td>
<td>6. CPCTC</td>
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EXAMPLE: Prove diagonals of a rectangle are congruent and bisect each other.

Given: Rectangle $ABDC$
Prove: $BC = AD$ and $BC$, $AD$ bisect each other

**PROOF**

1) $AC = BD$
   $AB = CD$
   **Definition of Rectangle**
   (opposite sides are congruent)
   (all angles are congruent, right angles)

2) $\angle CBD \cong \angle BCA$
   $\angle CAD \cong \angle BDA$
   **Parallel lines cut by a transversal, then alternate interior angles are congruent**

3) $\triangle ACM \cong \triangle DBM$
   **Congruent triangles Angle-Side-Angle**

4) $\frac{AM}{DM} = \frac{CM}{BM}$
   **CPCTC**

5) $BC$ and $AD$ bisect each other
   **Definition of Bisector**
   (A line, ray, or segment that cuts a segment into 2 congruent parts)

6) $\overline{CD} = \overline{CD}$
   **Reflexive Property**

7) $\angle C = \angle D = 90^\circ$
   $AC = BD$
   **Definition of Rectangle**

8) $\triangle ACD \cong \triangle BDC$
   **Congruent triangles Side-Angle-Side**

9) $\overline{BC} = \overline{AD}$
   **CPCTC**

Note: Pythagorean theorem can show that diagonals are equal
Proof Example
Proof of the "Parallelogram Diagonals Theorem"
(The Diagonals of a Parallelogram Bisect each other)

Given: Parallelogram
Prove: Diagonals Bisect Each Other

(Draw Picture)

(Establish Strategy)
Diagonals bisecting each other implies that congruent line segments are inside the parallelogram. (Also, notice that diagonals create triangles.)

(Label "things you know" about the
Given (Parallelogram))

PROOF

1) $AB = DC$
   $AD = BC$
   $AB \parallel BC$
   $AD \parallel BC$

   Definition of a Parallelogram
   (opposite sides are congruent)
   (opposite sides are parallel)

2) $\angle ACD \cong \angle CAB$
   $\angle ABD \cong \angle CDB$

   If two parallel lines are cut by a transversal, then alternate interior angles are congruent.

3) $\triangle ABE \cong \triangle CDE$

   ASA (Angle-Side-Angle)
   Triangles are congruent

4) $BE = DE$
   $AE = CE$

   CPCTC
   (Corresponding Parts of Congruent Triangles are Congruent)

5) AC bisects BD
   and
   BD bisects AC

   Definition of a Bisector
   (A line, ray, or segment that cuts a segment into 2 congruent parts)

("Diagonals of a Parallelogram bisect each other")
Prove that the **shortest** distance between a point and a line is a perpendicular line segment.

Step 1: Write out the Given and Prove statements

**Given:** Line AB with external point X  
Line segment XY is perpendicular to AB  
Segment XC is non-perpendicular to AB

**Prove:** Segment XY is shorter than segment XC

Step 2: Draw a diagram to clarify

![Diagram showing points A, C, Y, and B with X outside AB and angles XCY and XYC]

Step 3: Determine a strategy

We need to prove that $XY < XC$.
We see that we're dealing with a right triangle  
(angles X and C are both acute, and Y is 90 degrees)
And, we know that size of angles indicates size of opposite sides.

Step 4: Proof

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</table>
| 1. Line AB with external point X  
XY ⊥ AB | 1. Given |
| 2. $\angle XYC$ is a right angle | 2. Perpendicular lines form right angle |
| 3. $\angle XCY$ is an acute angle | 3. (Interior angles of Triangle add up to $180^\circ$)  
Non-right angles of a right triangle (i.e. $\triangle XYC$)  
are always acute ($< 90^\circ$) |
| 4. $\angle XCY < \angle XYC$ | 4. Definition of Acute Angle  
(Angles that are less than $90^\circ$) |
| 5. XY is shorter than XC | 5. In a triangle, the side opposite the smaller angle is shorter than a side opposite a larger angle  
$\angle XCY < \angle XYC$ so, $XY < XC$ |
**No-Choice Theorem Examples**

Given: \( \angle DBC \cong \angle E \)
Prove: \( \angle A \cong \angle BDC \)

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<td>2. Reflexive Property</td>
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<tr>
<td>3. ( \angle A \cong \angle BDC )</td>
<td>3. No-Choice Theorem</td>
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(If 2 angles of one triangle are congruent to 2 angles of another triangle, then the 3rd angles of both triangles are congruent)

Note: The angles are congruent.
So, the triangles are **similar**.
(We need at least one pair of congruent sides for congruent triangles)

---

Given: \( \angle ABC \cong \angle ACD \)
\( \angle ACB \cong \angle D \)

Are the triangles congruent?

Since two angles are congruent, the 3rd angles must be congruent (no-choice theorem)

We have angle-angle-angle...
(Similar Triangles)

BUT, the triangles may or may not be congruent...

**NOTE:** \( \overline{AC} \) in \( \triangle ABC \) does not correspond to \( \overline{AC} \) in \( \triangle ACD \)
"Noah's Arc"

"Perhaps you misunderstood the command to 'build an ark'?"

"I suppose I did..."

"...Hey, at least it floats."

Eventually, Noah realizes that this assignment was NOT a geometry construction
Thanks for visiting. (Hope it helped!)

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