Calculus: Introduction to Definite Integrals

**Fundamental Theorem of Calculus**

If a function $f$ is continuous on the interval $[a, b]$, then

$$\int_{a}^{b} f(x) \, dx = F(b) - F(a)$$

where $F$ is any function such that $F'(x) = f(x)$ for all $x$ in $[a, b]$.
Using Definite Integrals

A derivative determines the slope at a given point (or instantaneous rate of change). What can a definite integral do?

Answer: It can find area under a function over a specified interval.

Example:

\[
\int_{2}^{7} 4x + 6 \, dx
\]

Find the integral:

\[F(x) = \frac{4x^2}{2} + 6x = 2x^2 + 6x\]

Apply the Fundamental Theorem of Calculus:

\[F(7) = 2(7)^2 + 6(7) = 140\]
\[F(2) = 2(2)^2 + 6(2) = 20\]

Conclusion: The area under the function over the interval \([2, 7]\) is 120 (see graph).

Example:

\[
\int_{2}^{5} x^2 + 7 \, dx
\]

Find the integral:

\[F(x) = \frac{x^3}{3} + 7x + C\]  (indefinite integral)

Find the definite integral (apply the Fundamental theorem of Calculus):

\[
\left[\frac{x^3}{3} + 7x\right]_{2}^{5} = \left(\frac{5^3}{3} + 7(5)\right) - \left(\frac{2^3}{3} + 7(2)\right)
\]

\[= \frac{125}{3} + 35 - \frac{8}{3} - 14\]

\[= 60\]

(Notice: we can omit the constant 'C', because \(F(b) - F(a)\) would include \(C - C\), which would cancel the constant)

Fundamental Theorem of Calculus

If a function \(f(x)\) is continuous on the interval \([a, b]\), then

\[\int_{a}^{b} f(x) \, dx = F(b) - F(a)\]

where \(F(x)\) is any function such that \(F'(x) = f(x)\)

for all \(x\) in \([a, b]\)

Area of shaded region (trapezoid) = 120

The (blue) area under the function on the interval \([2, 5]\) has an area of 60 square units.
Graph and find the area bounds of the following:

1) \( y = |5 - x| \)
   and the x-axis
   on the interval [0, 8]

2) \( y = x + 3 \)
   \( y = -x + 6 \)
   x-axis and y-axis

3) \( y = e^x \)
   \( y = x \)
   \( x = 2 \)
   the y-axis
Evaluate the following definite integrals.
Then, sketch a graph, shading the area of the specified range.

$$\int_{1}^{4} (x^2 + 6) \, dx$$

$$\int_{3}^{5} (-x^2 + 7x - 10) \, dx$$

$$\int_{\frac{\pi}{2}}^{3\pi} \sin x + 3 \, dx$$
Fundamental Theorem of Calculus/Definite Integrals Exercise

Evaluate the definite integral. Then, sketch the function, shading the area of the specified range.

\[ \int_{7}^{14} \sqrt{x + 2} - 1 \, dx \]

Find the area bounded by \( x^2 - 4x - 5 \) and the x-axis. Sketch the function and label the area.

Find the total area enclosed by the x-axis and the cubic function

\[ f(x) = (x - 1)(x - 3)(x - 6) \]
Using Definite Integrals, find the shaded areas:

A)

B)

C)
“Suppose you have three Ford Mustangs. Inside each -- on the plush, leather seats -- is a six-pack of Coca-Cola....”

“How many cans of refreshing, ice cold Coke are there?”

“...and, the product of 6 and 3 is 18 cans of Coke.”

Multiplications Word Problems

\[
\frac{6 \text{ cans}}{\text{mustang}} \times 3 \text{ mustangs} = 18 \text{ cans}
\]

SLOPE: \( \frac{\text{rise}}{\text{run}} \)

Volume of Sphere: \( \frac{4}{3} \pi r^3 \)
Graph and find the area bounds of the following:

1) \( y = |5 - x| \)

and the x-axis
on the interval \([0, 8]\)

Since the shaded area consists of 2 right triangles, we can use simple area formula:

\[ A = \frac{1}{2} \text{(base)(height)} \]

\[ A_1 = \frac{1}{2} (5)(5) = 12.5 \]

\[ A_2 = \frac{1}{2} (3)(3) = 4.5 \]

Total = 17 sq. units

2) \( y = x + 3 \)

\( y = -x + 6 \)

x-axis and y-axis

First, find the intersection of the 2 lines:
(combination/elimination method)

\[ 2y = 9 \]
\[ y = 9/2 \]

so, \( x = 3/2 \)

Total: 15.75 sq. units

3) \( y = e^x \)

\( y = x \)

\( x = 2 \)

the y-axis

The shaded area consists of the area under the log function MINUS the right triangle (i.e. the area under the line \( y = x \))

(use definite integral to find area)

\[ A_{\log} = \int_{0}^{2} e^x \, dx = e^2 - e^0 = 7.39 - 1 \]

\[ A_{\text{tri}} = \frac{1}{2} (2)(2) = 2 \]

Total Area is approx. 4.39 sq. units
Fundamental Theorem of Calculus/Definite Integrals Exercise

Evaluate the following definite integrals. Then, sketch a graph, shading the area of the specified range.

\[ \int_{1}^{4} x + 6 \, dx = \left. \frac{x^2}{2} + 6x \right|_{1}^{4} = \left( \frac{4^2}{2} + 6(4) \right) - \left( \frac{1^2}{2} + 6(1) \right) = 8 + 24 - [\frac{1}{2} + 6] = 25 \frac{1}{2} \]

\[ \int_{3}^{5} -x^2 + 7x - 10 \, dx = \left. -\frac{x^3}{3} + \frac{7x^2}{2} - 10x \right|_{3}^{5} = \left( \frac{5^3}{3} - \frac{7(5)^2}{2} + 10(5) \right) - \left( \frac{3^3}{3} - \frac{7(3)^2}{2} - 10(3) \right) \]

\[ = \frac{-125}{3} + \frac{175}{2} - 50 - [-\frac{27}{3} + \frac{63}{2} - 30] = \frac{-125 + 175}{2} - \frac{85}{2} = 10 \frac{3}{2} \]

\[ \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin x + 3 \, dx = \left. -\cos x + 3x \right|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = -\cos \left( \frac{3\pi}{2} \right) + 3\left( \frac{3\pi}{2} \right) - \left( -\cos \left( \frac{\pi}{2} \right) + 3\left( \frac{\pi}{2} \right) \right) \]

\[ = 1 + \frac{9\pi}{2} - \left[ 0 + \frac{9\pi}{2} \right] = 1 + 3\pi \approx 10.4 \]
Evaluate the definite integral. Then, sketch the function, shading the area of the specified range.

\[ \int_{7}^{14} \sqrt{x + 2} - 1 \, dx \quad \int_{7}^{14} \frac{1}{(x + 2)^2} - 1 \, dx \]

\[ \frac{2}{3} (x + 2)^{3/2} - x \bigg|_{7}^{14} = \frac{2}{3} \left( (14 + 2)^{3/2} - 14 \right) - \left[ \frac{2}{3} \left( (7 + 2)^{3/2} - 7 \right) \right] \]

\[ = \frac{128}{3} - 14 - \left[ \frac{53}{3} \right] \quad = \frac{17}{3} \]

Find the area bounded by \( x^2 - 4x - 5 \) and the x-axis. Sketch the function and label the area.

first, find the zeros to determine the bounded range. Next, construct the definite integral. Then, evaluate.

(factors and set equal to 0): \( (x - 5)(x + 1) = 0 \) zeros: -1, 5

\[ \int_{-1}^{5} x^2 - 4x - 5 \, dx = \frac{x^3}{3} - 2x^2 - 5x \bigg|_{-1}^{5} \]

\[ = \frac{125}{3} - 50 - 25 - \left[ \frac{-1}{3} + 2 + 5 \right] = -36 \]

Find the total area enclosed by the x-axis and the cubic function \( f(x) = (x - 1)(x - 3)(x - 6) \)

Evaluate each part individually... then, combine the areas...

\( (x - 1)(x - 3)(x - 6) = x^3 - 10x^2 + 27x - 18 \)

\[ \int_{1}^{3} x^3 - 10x^2 + 27x - 18 \, dx = \frac{x^4}{4} - \frac{10x^3}{3} + \frac{27x^2}{2} - 18x \bigg|_{1}^{3} \]

\[ = \frac{81}{4} - 90 + \frac{243}{2} - 54 - \frac{1}{4} + \frac{10}{3} - \frac{27}{2} + 18 = 5.33 \]

\[ \int_{3}^{6} x^3 - 10x^2 + 27x - 18 \, dx = \frac{x^4}{4} - \frac{10x^3}{3} + \frac{27x^2}{2} - 18x \bigg|_{3}^{6} \]

\[ = 324 - 720 + 486 - 108 - \left[ \frac{81}{4} - 90 + \frac{243}{2} - 54 \right] = -15.75 \]

total area: 21.08 square units
Using Definite Integrals, find the shaded areas:

A) Identify the boundaries:
   -- the lower boundary: x-axis
   -- the upper boundary:
   for 0 ≤ x ≤ 1  y = x^2
   for 1 ≤ x ≤ 2  y = 2 - x

\[ \int_{0}^{1} x^2 \, dx + \int_{1}^{2} (2 - x) \, dx \]

\[ = \left[ \frac{x^3}{3} \right]_{0}^{1} + \left[ \frac{x^2}{2} \right]_{1}^{2} \]

\[ = \left( \frac{1}{3} - \frac{0}{3} \right) + \left( \frac{4}{2} - \frac{1}{2} \right) \]

\[ = \frac{1}{3} + 4 - 2 - \frac{1}{2} = \frac{5}{6} \]

B) Again, y = √x and y = x^2 meet at (1, 1)

Set up the integrals to find the area under the curves:

\[ \int_{0}^{1} \sqrt{x} \, dx + \int_{1}^{2} x^2 \, dx \]

\[ = \left[ \frac{x^{3/2}}{3/2} \right]_{0}^{1} + \left[ \frac{x^3}{3} \right]_{1}^{2} \]

\[ = \left( \frac{1}{3} - \frac{0}{3} \right) + \left( \frac{8}{3} - \frac{1}{3} \right) \]

\[ = \frac{2}{3} + \frac{8}{3} - \frac{1}{3} = 3 \]

C) Strategy: find area of entire rectangle and subtract the area under the curve.

Area of rectangle: lw
   length = π
   width = 2
   Area = 2π

Area under curve:
Find integral for 0 ≤ x ≤ π

\[ \int_{0}^{\pi} 1 + \cos x \, dx \]

\[ = [ x + \sin x ]_{0}^{\pi} \]

\[ = \pi + \sin \pi - (0 + \sin 0) \]

\[ = \pi - 0 - 0 - 0 = \pi \]

** Area of entire rectangle = 2π
Area under the curve = π

Therefore, shaded area is π.