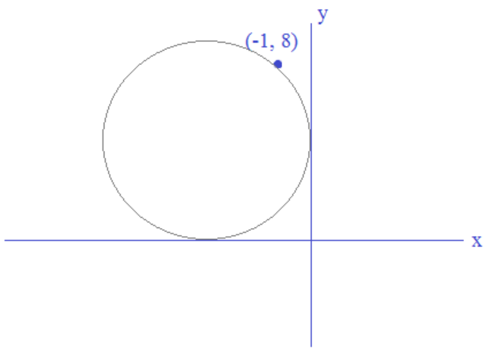
Conics VI: Honors and Analytic Geometry Examples

Topics include tangent lines, coordinate geometry, properties of conics, parametric equations, and more.

Mathplane.com

Example: Find the equation of a circle that is tangent to both axes and goes through the point $(-1, 8)$

Step 1: Draw a diagram



Step 2: Identify theorems and formulas that may help

Equation of a circle: $(x-h)^2 + (y-k)^2 = r^2$

where (h, k) is the center
and r is the radius

Geometry theorem: "all radii are congruent"

Distance Formula: $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
(optional)

Step 3: Apply concepts and theorems to set up equation

$$(x-h)^2 + (y-k)^2 = r^2$$

Because the circle is tangent to BOTH axes, the center will be some coordinate $(-r, r)$

Using substitution:

we'll take the point $(-1, 8)$ on the circle and the center $(-r, r)$

$$(-1-(-r))^2 + (8-r)^2 = r^2$$

$$(r-1)^2 + (8-r)^2 = r^2$$

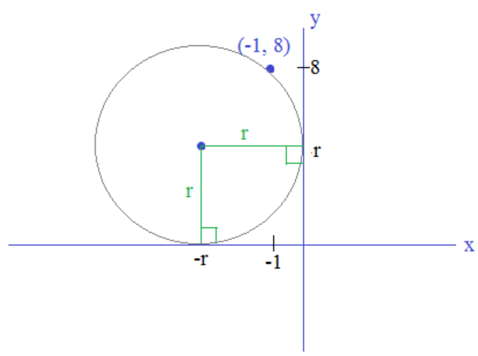
$$r^2 - 2r + 1 + 64 - 16r + r^2 = r^2$$

$$r^2 - 18r + 65 = 0$$

$$(r-5)(r-13) = 0$$

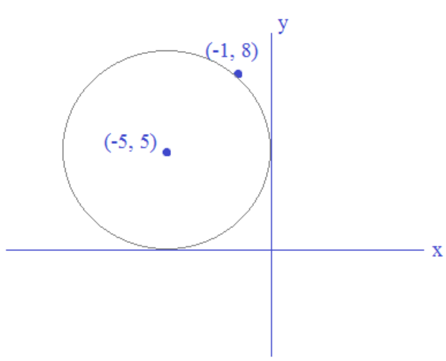
$$r = 5, 13$$

There are 2 circles!



Step 4: check answer

$(x+5)^2 + (y-5)^2 = 25$



$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

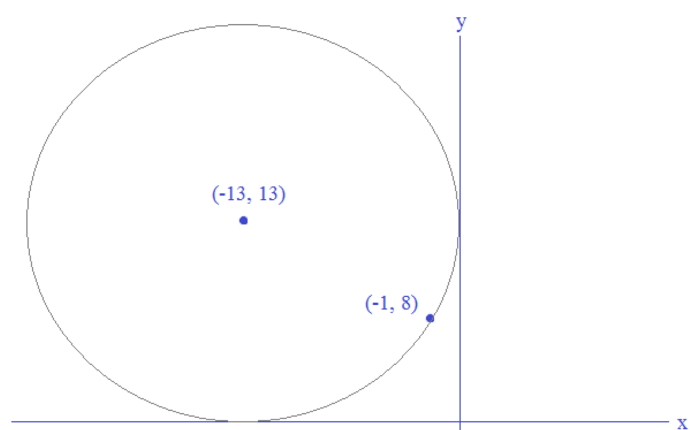
radius is $r = 5$

distance from $(-5, 5)$ to $(-1, 8)$ is

$$\sqrt{(-5+1)^2 + (5-8)^2}$$

$= 5$ ✓

$(x+13)^2 + (y-13)^2 = 169$



radius is $r = 13$

distance from $(-13, 13)$ to $(-1, 8)$ is

$$\sqrt{(-13+1)^2 + (13-8)^2}$$

$= 13$ ✓

Example: A parabola has a zero at 24 and a horizontal latus rectum of length 12 and an endpoint at (6, 9).

What is the equation of the parabola?

There are two possible answers.

1) latus rectum extends to the left from (6, 9) to (-6, 9)

Since latus rectum is horizontal, the parabola opens up/down..

$x^2 = 4py$

Length of latus rectum is 12, so p = 3

Focus is midpoint of LR: (0, 9)

We know (24, 0) is a point on the curve, so parabola would face down..

Vertex is (0, 12), because it is 3 units above the focus...

$x^2 = -12(y - 12)$

When we test the vertex (0, 12), the point works..

$(0)^2 = -12((12) - 12)$

$0 = 0$ ✓

When we test the point (24, 0), the point does NOT work...

$(24)^2 = -12((0) - 12)$

$576 = 144$ ✗

2) latus rectum extends to the right from (6, 9) to (18, 9)

Since latus rectum is horizontal, the parabola opens up/down..

$x^2 = 4py$

Length of latus rectum is 12, so p = 3

Focus is midpoint of LR: (12, 9)

Vertex: (12, 12)

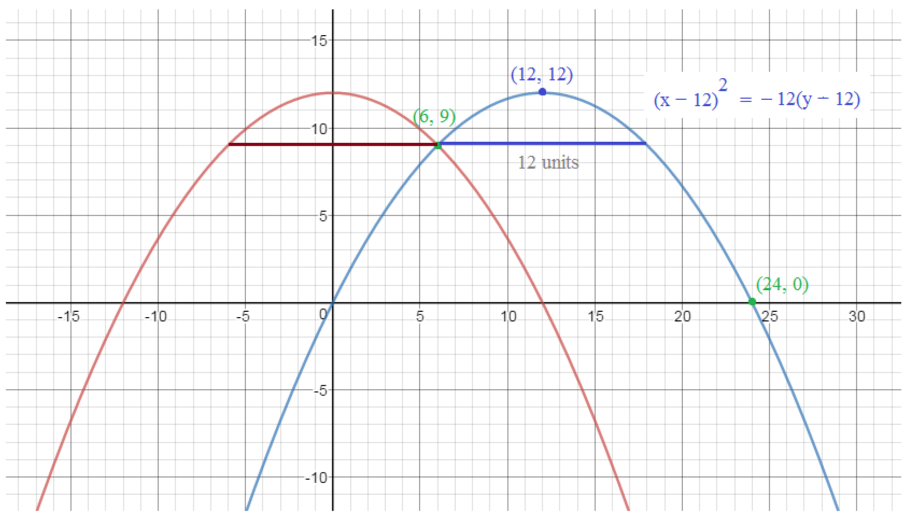
Directrix: y = 15

Parabola faces down, so p value is negative

$(x - 12)^2 = -12(y - 12)$

Does (24, 0) work?

Yes!!



Example: A circle that passes through (7, 3) and (5, 5) has its center on the line y = 4x + 1.

What is the equation of the circle?

(h, k) is the center

$k = 4h + 1$

$(x - h)^2 + (y - k)^2 = r^2$

Substitute each point:

$(7 - h)^2 + (3 - k)^2 = r^2$

$(5 - h)^2 + (5 - k)^2 = r^2$

$(7 - h)^2 + (3 - k)^2 = (5 - h)^2 + (5 - k)^2$

$k = 4h + 1$

$(7 - h)^2 + (3 - (4h + 1))^2 = (5 - h)^2 + (5 - (4h + 1))^2$

$49 - 14h + h^2 + 4 - 16h + 16h^2 = 25 - 10h + h^2 + 16 - 32h + 16h^2$

$49 - 14h + \cancel{h^2} + 4 - 16h + \cancel{16h^2} = 25 - 10h + \cancel{h^2} + 16 - 32h + \cancel{16h^2}$

$53 - 30h = 41 - 42h$

$h = -1$ then, $k = -3$

$(x + 1)^2 + (y + 3)^2 = 100$

to check and find radius, determine the distance from center to each point...

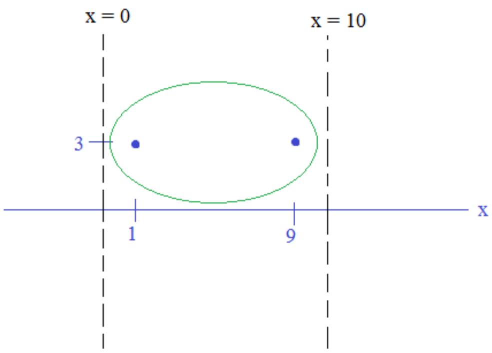
(-1, -3) to (5, 5) ----> 10

(-1, -3) to (7, 3) ----> 10

Example: Write the equation of an ellipse with foci at (1, 3) and (9, 3) and directrices of $x = 0$ and $x = 10$

The Ellipse Directrix

Step 1: Draw a sketch with given information



Step 2: Use formulas and equations

Directrix: $\dfrac{a^2}{c}$ $\qquad$ $a^2 - b^2 = c^2$

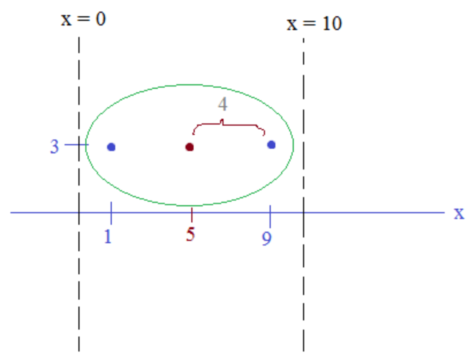
Center is midpoint of foci

Equation of Ellipse: $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

Step 3: Start filling in parts

center is (5, 3) $\qquad \dfrac{(x-5)^2}{a^2} + \dfrac{(y-3)^2}{b^2} = 1$

distance from center to each focus is $c = 4$



Directrix: $\dfrac{a^2}{c}$ ----> distance from center to directrix

$$\frac{a^2}{4} = 5$$

$$a^2 = 20$$

$$\frac{(x-5)^2}{20} + \frac{(y-3)^2}{b^2} = 1$$

$$\frac{(x-5)^2}{20} + \frac{(y-3)^2}{4} = 1$$

$$a^2 - b^2 = c^2$$

$$20 - b^2 = 16$$

$$b^2 = 4$$

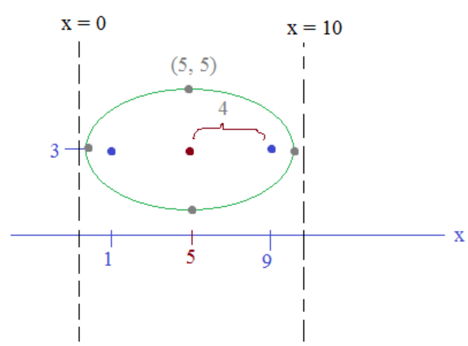
center: (5, 3)

foci: (1, 3) and (9, 3)

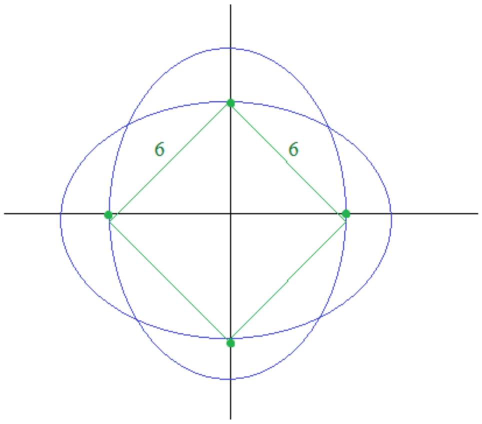
vertices: $(5 + \sqrt{20}, 3)$ and $(5 - \sqrt{20}, 3)$

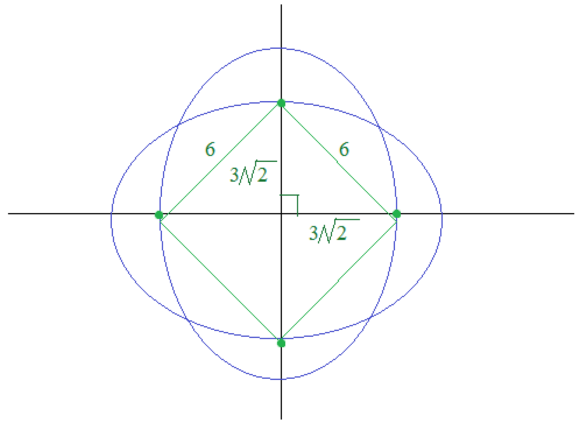
covertices: (5, 5) and (5, 1)

directrices: $x = 0$ and $x = 10$



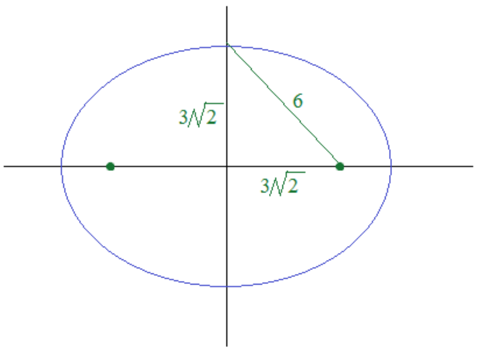
Examples: An ellipse with major axis parallel to the x-axis intersects another ellipse with major axis parallel to the y-axis. Each ellipse passes through the foci of the other ellipse, which form the vertices of a square. If the square has area of 36, what is the area enclosed by one of the ellipses?





Foci: $(3\sqrt{2}, 0)$ and $(-3\sqrt{2}, 0)$

Co-Vertices: $(0, 3\sqrt{2})$ and $(0, -3\sqrt{2})$



Area of an ellipse $= AB\pi$

$= (6)(3\sqrt{2})\pi$

$= 18\sqrt{2}\pi$

$C = 3\sqrt{2}$

$B = 3\sqrt{2}$

$$\frac{x^2}{36} + \frac{y^2}{18} = 1$$

$C^2 = A^2 - B^2$

$3\sqrt{2}^2 = A^2 - 3\sqrt{2}^2$

$18 = A^2 - 18$

$A = 6, -6$

Example: $x_1 = 3\cos t$ $\quad x_2 = -3 + \cos t$

$y_1 = 2\sin t$ $\quad y_2 = 1 + \sin t$

a) Graph the conics, and determine their intersection...

b) If the path of a particle (x_1, y_1) is the first equation, and the path of a particle (x_2, y_2) is the second equation, do the particles collide?

Removing the Parameter to graph

$x_1 = 3\cos t \qquad y_1 = 2\sin t$

$\cos t = \frac{x}{3} \qquad \sin t = \frac{y}{2}$

$\sin^2 + \cos^2 = 1$

$\left(\frac{y}{2}\right)^2 + \left(\frac{x}{3}\right)^2 = 1$

$\frac{x^2}{9} + \frac{y^2}{4} = 1$

$x_2 = -3 + \cos t \qquad y_2 = 1 + \sin t$

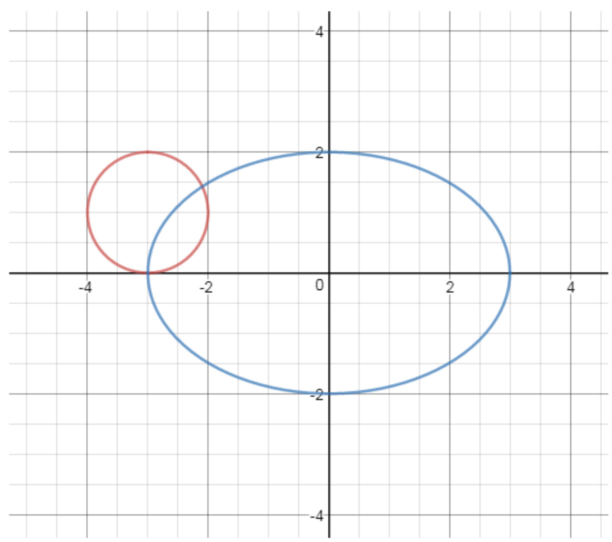
$\cos t = x + 3 \qquad \sin t = y - 1$

$\sin^2 + \cos^2 = 1$

$(y-1)^2 + (x+3)^2 = 1$

intersection:

(-3, 0) and (-2.1, 1.4)



$x_1 = 3\cos t \qquad x_2 = -3 + \cos t$

$y_1 = 2\sin t \qquad y_2 = 1 + \sin t$

Set the x's and y's equal to each other

$3\cos t = -3 + \cos t \qquad 2\sin t = 1 + \sin t$

$2\cos t = -3 \qquad \sin t = 1$

$\cos t = -3/2 \qquad t = 90^0$

no solution

at $t = 90^0$

$x_1 = 3(0) \qquad x_2 = -3 + (0)$

$y_1 = 2(1) \qquad y_2 = 1 + (1)$

(0, 2) $\qquad$ (-3, 2)

particles will never collide...

$x_1 = 3\cos t$

$y_1 = 2\sin t$

at (-3, 0) ----> $3\cos t = -3$

$2\sin t = 0$

$t = 180^\circ$ (or π)

$x_2 = -3 + \cos t$

$y_2 = 1 + \sin t$

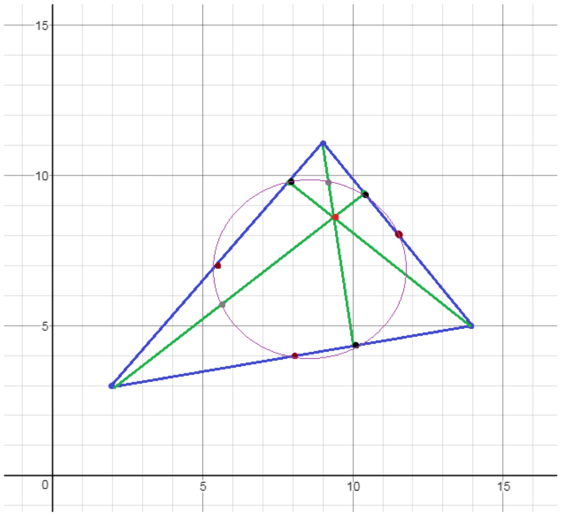
if $t = 180^\circ$ (or π) then $-3 + \cos(180^\circ) = -4$

$1 + \sin(180^\circ) = 1$

(-4, 1)

"The 9-Points Circle"

Example: Find the 9-points circle of a triangle with vertices (2, 3) (9, 11) and (14, 5)



Determine the midpoint of each side...

(5.5, 7) (11.5, 8) (8, 4)

The midpoints are 3 points of the circle...

Draw the 3 altitudes...

The base of the altitudes are 3 points of the circle...

The orthocenter is the intersection of the 3 altitudes..

Identify the midpoint between the 3 vertices and the orthocenter

The 3 vertex/orthocenter midpoints are 3 points of the circle...

***Note: To find the orthocenter (exactly), you must find the intersection of 2 altitudes...
(In other words, find equation of 2 altitudes, then use system of linear equations to find intersection)

(2, 3) and (14, 5) slope: 2/12 = 1/6

⇒ slope of (perpendicular) altitude: -6

altitude runs through point (9, 11)

$y - 11 = -6\,(x - 9)$

(9, 11) and (14, 5) slope: -6/5

⇒ slope of altitude: 5/6

altitude runs through point (2, 3)

$y - 3 = \frac{5}{6}\,(x - 2)$

system of equations ----> orthocenter: of triangle (9.32, 9.10)

Quick note: The orthocenter in the graph appears to be around (9.4, 8.7)... This difference is due to the graph being a "hand sketch", so the altitudes may not be "exact"...

circle: $(x - h)^2 + (y - k)^2 = r^2$

Finding circle that circumscribes the triangle

(2, 3) (9, 11) and (14, 5)

$(2 - h)^2 + (3 - k)^2 = r^2$

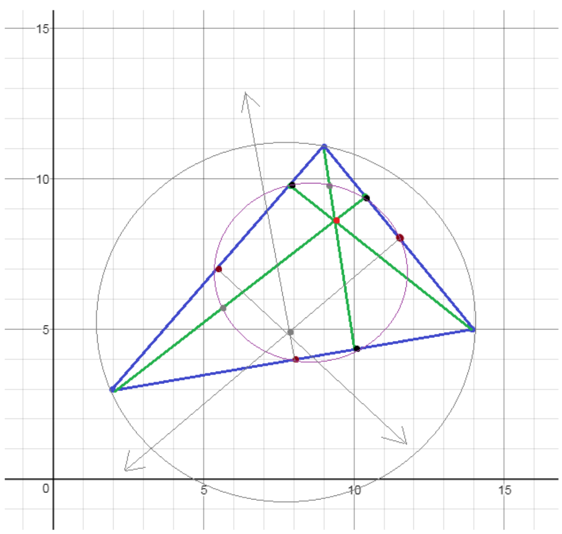
$(9 - h)^2 + (11 - k)^2 = r^2$

$(14 - h)^2 + (5 - k)^2 = r^2$

after substituting each point on the circle into the standard form of a circle, we have 3 equations with 3 unknowns...

h = 7.84 k = 4.95 r = +6.16 or ~~-6.16~~

center: (7.84, 4.95) radius: 6.16



NOTE: radius of 9 points circle is HALF the radius of the circle that circumscribes the triangle

To find the circle that circumscribes a triangle:
Draw 3 perpendicular bisectors from the sides
Find the intersection of the 3 perpendicular bisectors (circumcenter)
**The circumcenter is equidistant to the 3 vertices of the triangle!
Therefore, it is the center of the circumscribed circle..

3 gray perpendicular bisectors...

the gray intersection is the circumcenter...

the gray large circle is circumscribed around the triangle..

approx. radius is 6.2

therefore, the radius of 9-points circle would be approx. 3.1

Finding equation of 9-points circle

circle: $(x - h)^2 + (y - k)^2 = r^2$

(5.5, 7) (11.5, 8) (8, 4)

(using the 3 midpoints of the triangle sides)

after substituting each point on the circle into the standard form of a circle, we have 3 equations with 3 unknowns...

Equation of 9 points circle: we know the radius is approx. 3.1 (half of circumscribed circle) so, we need the center...

$(5.5 - h)^2 + (7 - k)^2 = r^2$

$(11.5 - h)^2 + (8 - k)^2 = r^2$

$(8 - h)^2 + (4 - k)^2 = r^2$

h = 8.57

k = 7.02

r = 3.08 or ~~-3.08~~

center: (8.57, 7.02)

radius: 3.08

Example: Find an equation of the line tangent to $y = x^2$ at (1, 1)

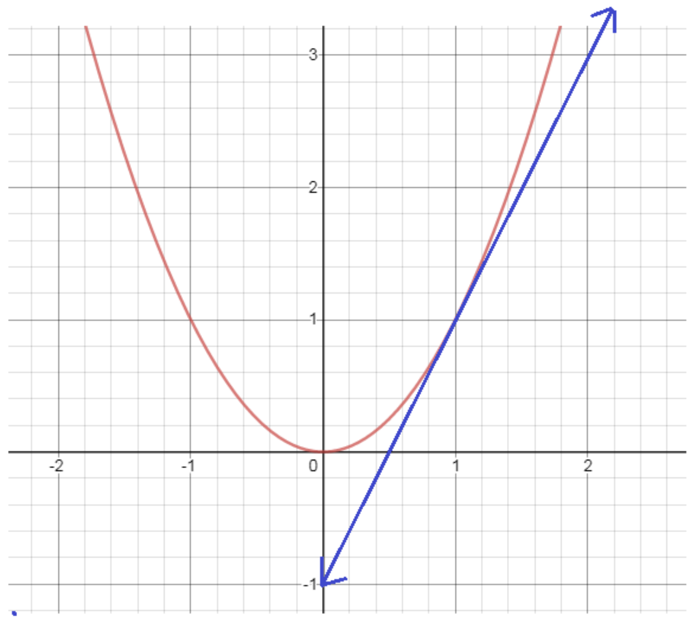
Using Calculus:

$$y = x^2$$

the derivative $y' = 2x$

therefore, the slope of the tangent line at $x = 1$ is 2

equation of tangent line: $y - 1 = 2(x - 1)$



Using Analytic Geometry:

Equation of a line: $y - 1 = m(x - 1)$

$\Rightarrow y = mx - m + 1$

Equation of parabola:

$\Rightarrow y = x^2$

Now, let's solve the system!

$$mx - m + 1 = x^2$$

Rearrange....

$$x^2 - mx + (m - 1) = 0$$

NOTE: A tangent line has only one point of intersection. (i.e. one solution)

If you drew a line that intersected (1, 1) that was NOT tangent, it would have 2 solutions!
(2 intersections)

Since we are looking for ONE solution, this quadratic must yield a discriminant

$$B^2 - 4AC = 0$$

$A = 1$

$B = -m$

$C = m - 1$

$$(-m)^2 - 4(1)(m - 1) = 0$$

$$m^2 - 4m + 4 = 0$$

$$(m - 2)^2 = 0$$

slope $m = 2$

$$y - 1 = 2(x - 1)$$

Example: Find the equation of the parabola with focus (0, 0) and directrix $x + y = 4$

First, let's find the vertex....

We know the vertex is equidistant from the focus and directrix....

focus: (0, 0)

directrix: $y = -x + 4$

Using geometry, we can determine the vertex...

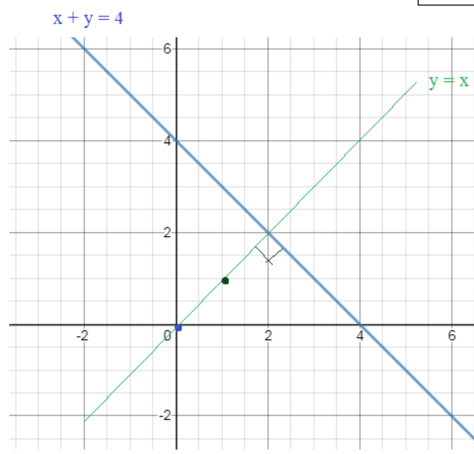
Since directrix slope is -1, we know the slope of a perpendicular line is 1...

and, since the perpendicular line goes through (0, 0), we have $y = x$

then, solving system of equations, we know the intersection of

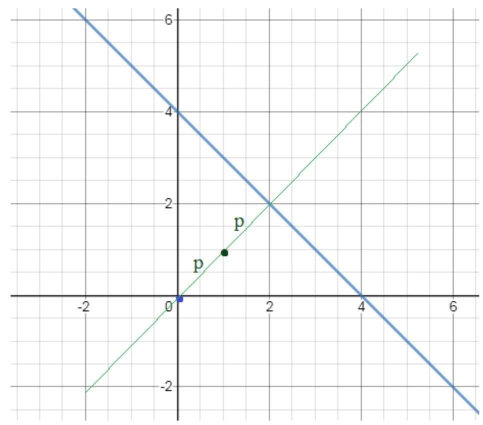
$y = x$ and $y = -x + 4$ is (2, 2)

The intersection is (2, 2).... therefore, the midpoint between the focus (0, 0) and the intersection (2, 2) is (1, 1)



We now know the vertex is (1, 1)....

and, we see that the directrix has been rotated 45 degrees....



Let's remember the definition of a parabola....

distance p from vertex to directrix (or focus) is

$$p = \sqrt{(1-2)^2 + (1-2)^2} = \sqrt{2}$$

***definition of parabola: any point on the parabola is equidistant to the focus and directrix!

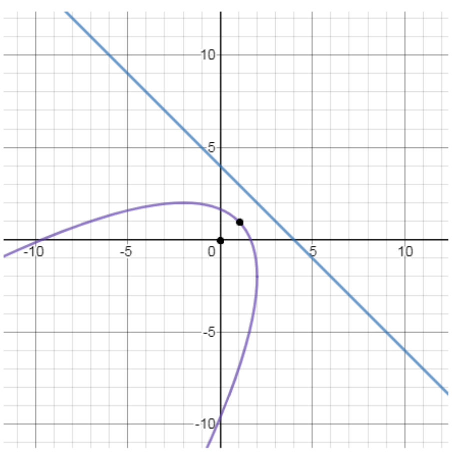
$x + y = 4$ -----> $x + y - 4 = 0$ A = 1, B = 1, C = -4

Distance from a point to a line: (directrix)

$$\frac{Ax + By + C}{\sqrt{A^2 + B^2}} \text{ -----> } \frac{x + y + (-4)}{\sqrt{1^2 + 1^2}}$$

Distance from a point to the focus: focus: (0, 0)

$$\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$$



Distance from any point to the directrix — Distance from any point to the focus

$$\frac{x + y + (-4)}{\sqrt{1^2 + 1^2}} = \sqrt{x^2 + y^2}$$

cross multiply....

$$\sqrt{2x^2 + 2y^2} = x + y - 4$$

square both sides...

$$2x^2 + 2y^2 = (x + y - 4)^2$$

$$2x^2 + 2y^2 = x^2 + xy + (-4x) + xy + y^2 + (-4y) - 4x - 4y + 16$$

collect like terms...

$$x^2 + y^2 = 2xy - 8x - 8y + 16$$

$$x^2 - 2xy + y^2 + 8x + 8y - 16 = 0$$

Example: *Example:* Find the equation of the line(s) tangent to $y^2 = 2(x-5)$ that pass through the point (2, 1)

The equation of a line passing through (2, 1) will be $y - 1 = m(x-2)$

$y = mx - 2m + 1$ where m is the slope of the line...

And, the equation of the parabola is $y^2 = 2x - 10$

To find the intersection of the line and parabola, we will solve the system.....

$$y = mx - 2m + 1$$
$$y^2 = 2x - 10$$

using substitution: $(mx - 2m + 1)^2 = 2x - 10$

expand

$$m^2x^2 - 2m^2x + mx - 2m^2x + 4m^2 - 2m + mx - 2m + 1 = 2x - 10$$

combine "like" terms

$$m^2x^2 - 4m^2x + 2mx + 4m^2 - 4m + 1 = 2x - 10$$

$$m^2x^2 - 4m^2x + 2mx - 2x + 4m^2 - 4m + 11 = 0$$

separate/factor the coefficients

$$(m^2)x^2 + (-4m^2 + 2m - 2)x + (4m^2 - 4m + 11) = 0$$

A B C

Since the intersection of a tangent line is a unique point, the quadratic discriminant

$$B^2 - 4AC = 0$$

$$(-4m^2 + 2m - 2)^2 - 4(m^2)(4m^2 - 4m + 11) = 0$$

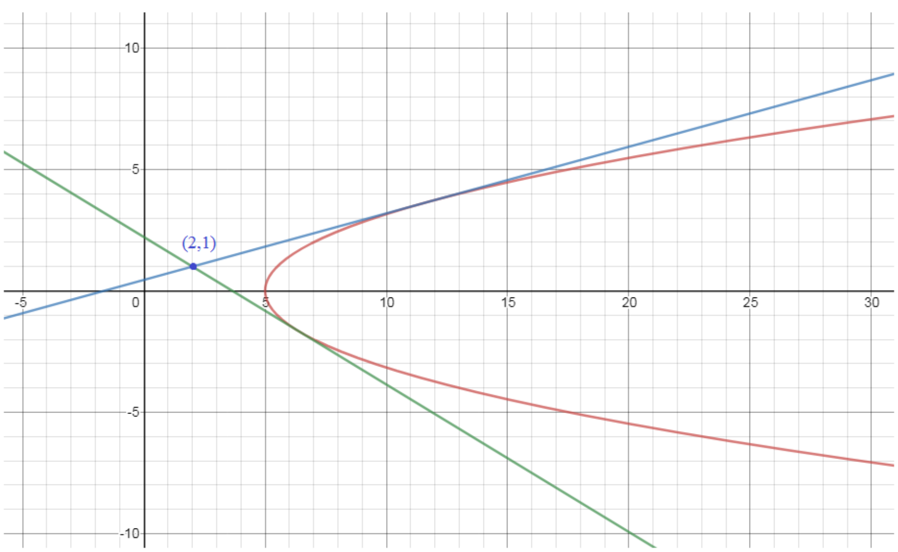
solve (with calculator)

$$m = \frac{\sqrt{7} - 1}{6} \qquad m = \frac{-\sqrt{7} - 1}{6}$$

approx. = .2743 approx. = -0.6076

Therefore, the lines passing through the point (2, 1) would be

$y - 1 = .2743(x-2)$ and $y - 1 = -0.6076(x-2)$



Example: Find the equation of the line tangent to $5x^2 + 3y^2 = 32$ @ (1, 3)

Using Analytic Geometry:

Equation of a line: $y - 3 = m(x - 1)$

$\Rightarrow$ $y = mx - m + 3$

Equation of ellipse:

$\Rightarrow$ $5x^2 + 3y^2 = 32$

Now, let's solve the system! (using substitution)

$$5x^2 + 3(mx - m + 3)^2 = 32$$

$$(mx - m + 3)^2$$

$$(mx - m + 3)(mx - m + 3)$$

$$\begin{array}{l} m^2x^2 - m^2x + 3mx \\ \quad - m^2x \quad + m^2 - 3m \\ \quad\quad 3mx \quad - 3m + 9 \\ \hline m^2x^2 - 2m^2x + 6mx + m^2 - 6m + 9 \end{array}$$

$$5x^2 + 3(m^2x^2 - 2m^2x + 6mx + m^2 - 6m + 9) = 32$$

$$5x^2 + 3m^2x^2 - 6m^2x + 18mx + 3m^2 - 18m + 27 = 32$$

Rearrange....

$$(5 + 3m^2)x^2 + (18m - 6m^2)x + (3m^2 - 18m - 5) = 0$$

$Ax^2 + Bx + C$

$A = 5 + 3m^2$

$B = 18m - 6m^2$

$C = 3m^2 - 18m - 5$

Since we are looking for ONE solution, this quadratic must yield a discriminant

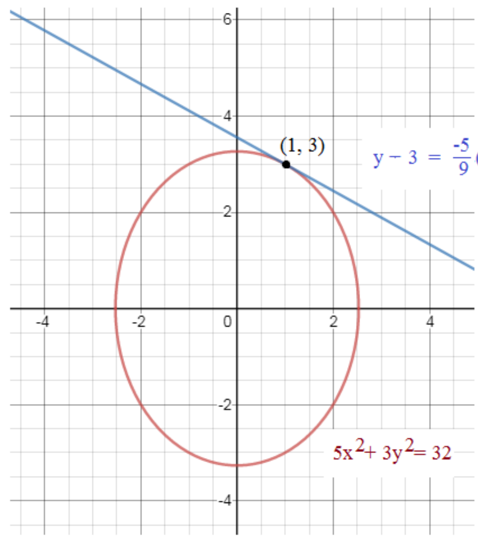
$$B^2 - 4AC = 0$$

$$(18m - 6m^2)^2 - 4(5 + 3m^2)(3m^2 - 18m - 5) = 0$$

$$m = -5/9$$

Equation of line passing through (1, 3)

$$y - 3 = \frac{-5}{9}(x - 1)$$



Using Calculus: $5x^2 + 3y^2 = 32$

the derivative $10x + 6y\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-10x}{6y}$$

therefore, the slope of the tangent line at (1, 3) $\Rightarrow \frac{-10}{18}$

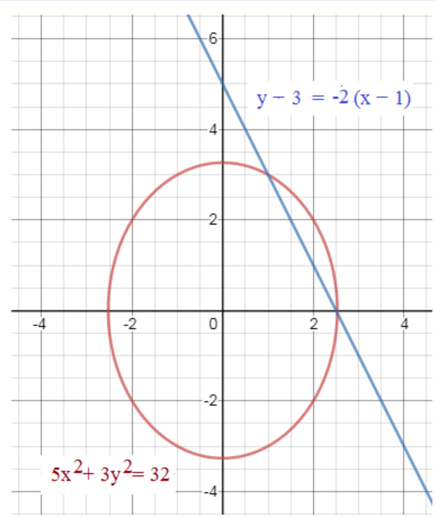
equation of tangent line: $y - 3 = \frac{-5}{9}(x - 1)$

NOTE: A tangent line has only one point of intersection. (i.e. one solution)

If you drew a line that intersected (1, 3) that was NOT tangent, it would have 2 solutions!
(2 intersections)

$m = -2$

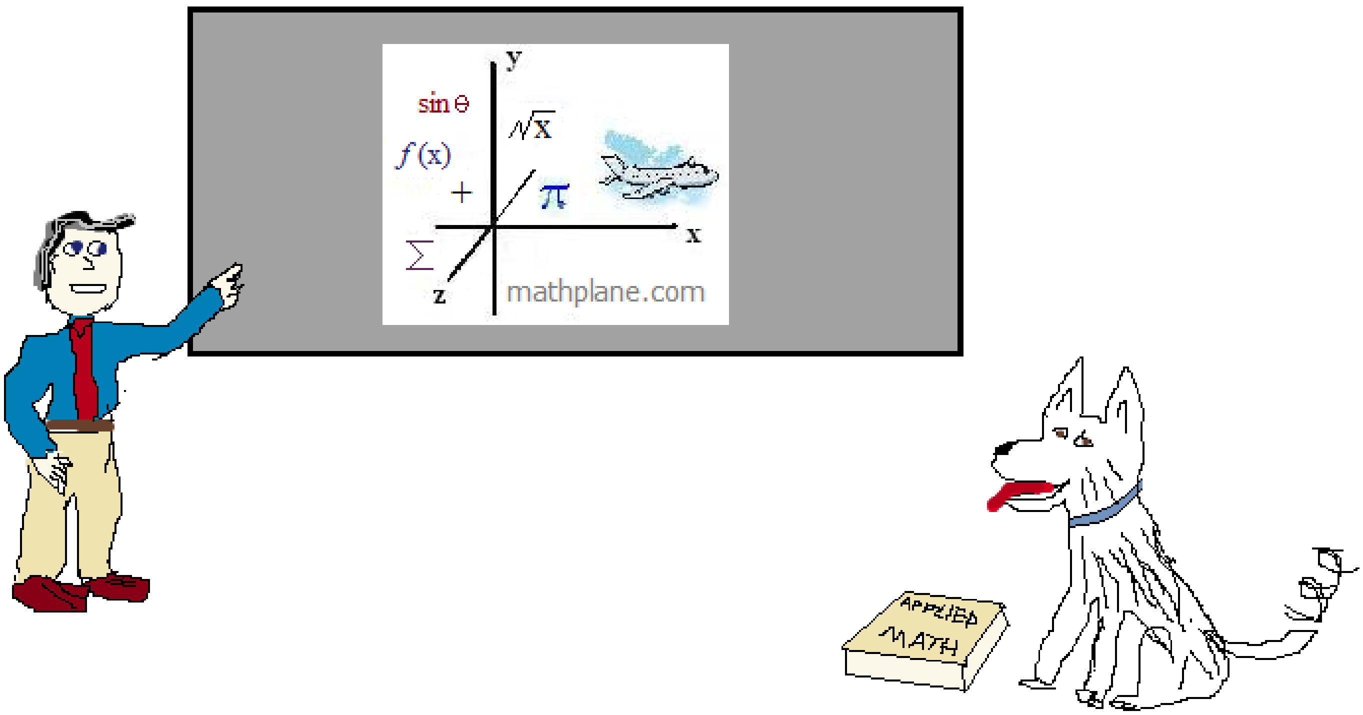
$B^2 - 4AC \neq 0$



Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers



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